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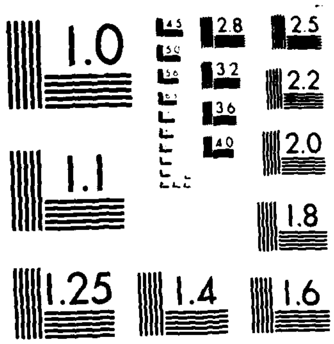
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DISSERTATION

NONLINEAR METHODS FOR SPACECRAFT ATTITUDE MANEUVERS

Submitted by

Major Albert L. Batten, USAF

Department of Electrical Engineering

In partial fulfillment of the requirements

for the Degree of Doctor of Philosophy

Colorado State University

Fort Collins, Colorado

Summer 1985

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ABSTRACT OF DISSERTATION

NONLINEAR METHODS FOR SPACECRAFT ATTITUDE MANEUVERS

Spacecraft attitude control is described by a nonlinear dynamic mathematical model. When Euler parameters are used to define the orientation of a spacecraft with respect to an inertial frame, the model takes ~~the~~ a certain form $\dot{\underline{\xi}} = \underline{\phi}(\underline{\xi}) + \Psi \underline{\sigma}$ where the components of are factor $\underline{\phi}$ are quadratic or higher order functions of the state variables. Large-angle maneuvers have traditionally been accomplished as a succession of small-angle rotations from a sequence of defined operating points about which the model is linearized. Alternativately, a sequence of single axis rotations are performed. These processes are computationally burdensome and are unable to produce fast multi-axial large-angle rotational maneuvers.

State-dependent linearizing transformations allow an equivalent linear system of the form $\dot{\underline{x}}_1 = \underline{x}_2$, $\dot{\underline{x}}_2 = -a_o\underline{x}_1 - a_1\underline{x}_2 + \underline{u}$ to be defined where a_o and a_1 are freely chosen. A control strategy $\underline{u}(t)$ is determined which forces the linear equivalent system to evolve in a desired manner. The input $\underline{\sigma}$ to the physical system is then retrieved by an inverse transformation. Advantages of this approach to spacecraft attitude control include the availability of linear systems techniques in control design, realizability of fast multi-axial large-angle maneuvers, and with appropriate choice of performance criteria, the ability to achieve exact terminal states by specified times.

Imperfect knowledge of the spacecraft inertial properties leads to significant perturbations from the terminal state. Introduction of a linear regulator acting on the state errors in the linear model corrects for these perturbations. Commanded torques generated by solution of

unconstrained optimization problems for the linear model may cause saturation of torque actuators. A technique of reinitializing the control strategy upon encountering saturation is introduced.

COLORADO STATE UNIVERSITY

Summer 1985

WE HEREBY RECOMMEND THAT THE THESIS PREPARED UNDER OUR SUPERVISION

BY Major Albert L. Batten

ENTITLED NONLINEAR METHODS FOR SPACECRAFT ATTITUDE MANEUVERS

BE ACCEPTED AS FULFILLING IN PART REQUIREMENTS FOR THE DEGREE OF

DOCTOR OF PHILOSOPY.

Committee on Graduate Work

David W. Zachmann

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ABSTRACT OF DISSERTATION

NONLINEAR METHODS FOR SPACECRAFT ATTITUDE MANEUVERS

Spacecraft attitude control is described by a nonlinear dynamic mathematical model. When Euler parameters are used to define the orientation of a spacecraft with respect to an inertial frame, the model takes the form $\dot{\underline{\xi}} = \underline{\phi}(\underline{\xi}) + \mathcal{W}\underline{\sigma}$ where the components of $\underline{\phi}$ are quadratic or higher order functions of the state variables. Large-angle maneuvers have traditionally been accomplished as a succession of small-angle rotations from a sequence of defined operating points about which the model is linearized. Alternativately, a sequence of single axis rotations are performed. These processes are computationally burdensome and are unable to produce fast multi-axial large-angle rotational maneuvers.

State-dependent linearizing transformations allow an equivalent linear system of the form $\dot{\underline{x}}_1 = \underline{x}_2$, $\dot{\underline{x}}_2 = -a_o\underline{x}_1 - a_1\underline{x}_2 + \underline{u}$ to be defined where a_o and a_1 are freely chosen. A control strategy $\underline{u}(t)$ is determined which forces the linear equivalent system to evolve in a desired manner. The input $\underline{\sigma}$ to the physical system is then retrieved by an inverse transformation. Advantages of this approach to spacecraft attitude control include the availability of linear systems techniques in control design, realizability of fast multi-axial large-angle

maneuvers, and with appropriate choice of performance criteria, the ability to achieve exact terminal states by specified times.

A singularity condition exists in the inverse transformation when detumbling of the spacecraft is desired. Sufficient conditions are developed under which the singularity will not be excited. Imperfect knowledge of the spacecraft inertial properties leads to significant perturbations from the terminal state. Introduction of a linear regulator acting on the state errors in the linear model corrects for these perturbations. Commanded torques generated by solution of unconstrained optimization problems for the linear model may cause saturation of torque actuators. A technique of reinitializing the control strategy upon encountering saturation is introduced. Simulations illustrate the feasibility of the described approaches to spacecraft attitude control.

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Above all else, I must acknowledge the Lord God, who created the universe, designed the natural laws that it should obey, and gave mankind wisdom to understand.

Fort Collins, CO A. L. Batten

April 1985

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CHAPTER ONE

INTRODUCTION

1.1 Background

As the interest in, and application of, space vehicles increase in magnitude and complexity, so the requirements of accuracy and timeliness of control strategies for those space vehicles also increase. Our country's present administration has proposed researching the feasibility of a space-based defense system which would significantly reduce our vulnerability to long-range missile attack. This latest proposal serves to highlight the advances that have already been made in space technology and the continuation of efforts to use this technology to our benefit.

An example of current interest is the Venus Radar Mapping Mission planned for early 1986 [1]. The objective of this interplanetary mission is to precisely map surface characteristics of Venus. The spacecraft will be maneuvered into an elliptical orbit about the planet. During the multiple close passes, several degrees either side of pericention, radar images of the surface will be constructed. As the satellite moves toward the apocention of the orbit, it will be repositioned to transmit these images to receiving stations on Earth. The data gathering and transmitting sequences will require extremely accurate repositioning, or reorientation, of the spacecraft.

Department of Defense interest in high performance spacecraft attitude control is evidenced in co-sponsorship by the Air Force Office of Scientific Research, with NASA Langley Research Center, of annual workshops on dynamics and control of large space structures. One of the issues emphasized is fast maneuvering of these spacecraft.

> "The new algorithm research has been driven by requirements for spacecraft control software that is relatively simple, robust, and capable of being executed by prospective on-board computers under demanding operational and time constraints... Such spacecraft constitute large, relatively flexible structures that must slew quickly through large angles, then rapidly orient and stabilize their attitudes to enable their on-board acquisition, pointing, and tracking (APT) systems to acquire their next target." [2]

The objective of current research in spacecraft attitude control is perhaps best summed up in comments made by Lt. Gen. James A. Abrahamson, Director of the Space Defense Initiative, to Aviation Week and Space Technology.

> "...; what we have to do is move our own system very quickly a small angular distance and then stabilize and shoot — so our job is to move quickly, stabilize and knock it out and move on to the next." [3]

System equations for multi-axial reorientation of spacecraft are nonlinear. If the attitude is expressed in terms of Euler angles, then transcendental functions appear throughout the model. When the spacecraft orientation is defined by Euler parameters, the system equations are functions of quadratic terms of the state variables. A standard approach to dealing with this nonlinear model is to define an operating point about which the model is linearized. Small displacements from this nominal operating point can be effected with

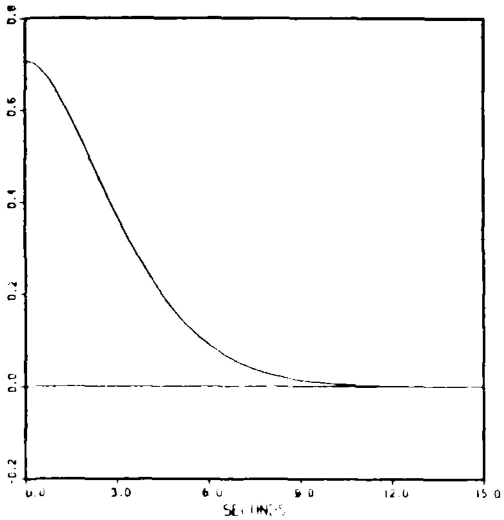
little error. The magnitude of the displacements that are allowed is governed by the error tolerance acceptable for the problem.

For a multi-axial large-angle maneuver to be completed, successive operating points are defined, small displacements are commanded, and progress along a prescribed nominal path is monitored by on-board sensors. The result can be a quite accurate reorientation, but this process is very slow and computationally burdensome.

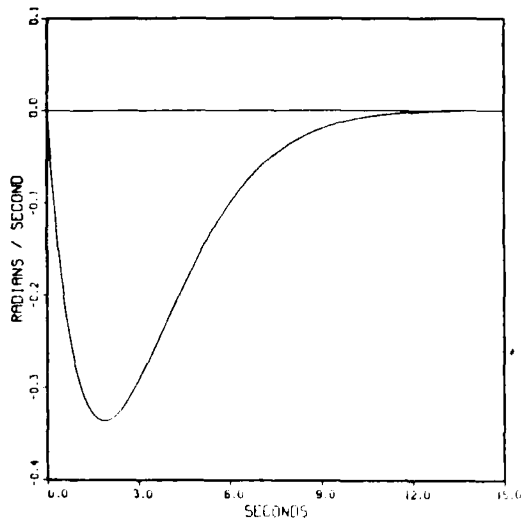
Single axis maneuvers may be represented by linear systems models. An alternative to the linearization process described is a sequence of single axis maneuvers where only one reaction wheel or thruster pair is used for each maneuver. A time-consuming reorientation is again the result.

To circumvent the slow successive linearization or sequential single-axis processes, Junkins and Turner [4] formulated a nonlinear optimization problem. They proposed a relaxation process for determining costate initial values which then allow numerical solution of the state and costate differential equations. This iterative process must be performed for each desired set of initial and final state values. Solutions determined by this approach converged with an accuracy determined by the specific computer in use.

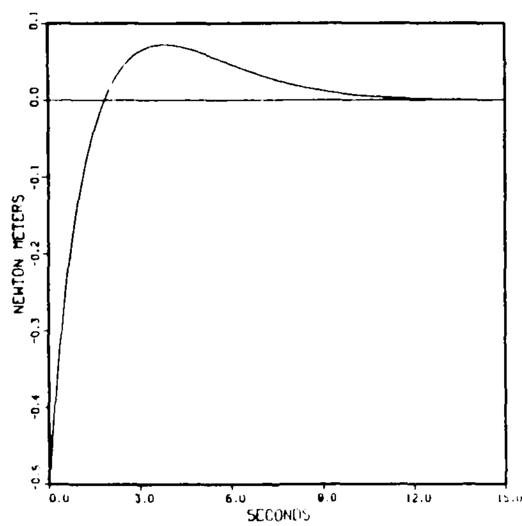
Vadali and Junkins [5] suggested using the method of particular solutions to solve the two point boundary problem for state and costate histories. As presented in their paper, this method makes use of quasilinearization, (linearization about a nominal trajectory instead of about a nominal operating point). Again an iterative process is employed to converge to the desired solution, and storage of the entire numerical costate trajectories is required.



(a) Attitude Parameter (γ)

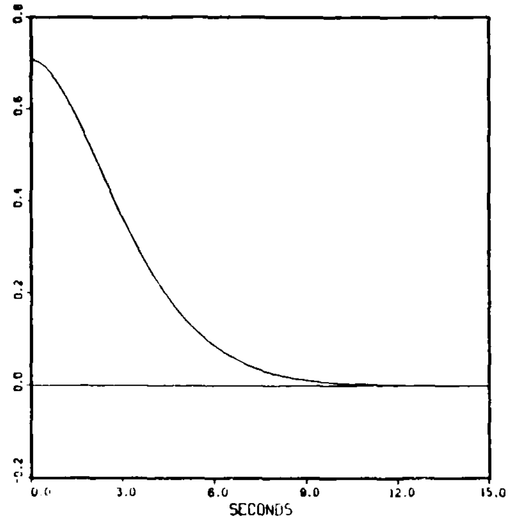


(b) Angular Velocity (ω)



(c) Applied Torque (τ)

Figure 1.4. Simulation of single-axis reorientation using only linear and cubic feedback terms.



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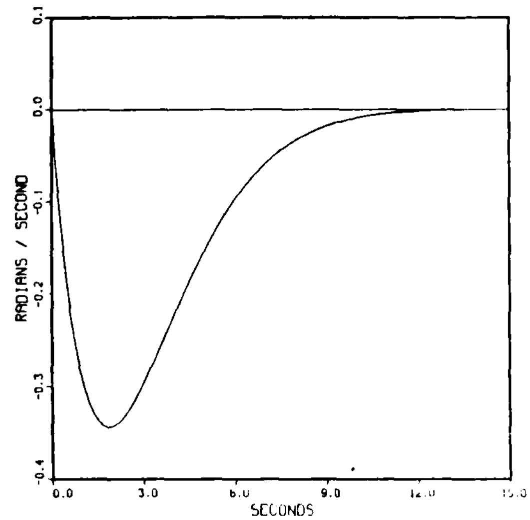
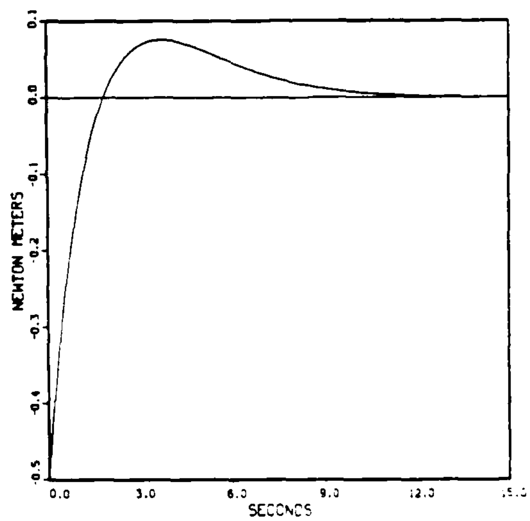


(b) Angular Velocity (ω)



(c) Applied Torque (τ)

Figure 1.3. Simulation of single-axis reorientation using only linear feedback terms.

$$\tau = [\delta_1\gamma + \delta_2\omega] + [\delta_3\gamma^3 + \delta_4\gamma^2\omega + \delta_5\gamma\omega^2 + \delta_6\omega^3].$$

The first two coefficients are given by $\delta_1 = \sqrt{q_1}$ and $\delta_2 = -\sqrt{I_o\sqrt{q_1} + q_2}$. Expressions for the four third order coefficients depend on the closed loop transition matrix for the linear portion of the state equations. They are lengthy and extremely complex, however they are achievable with the use of simple algebra and much determination (see App. A).

Simulations were run for I_o=1.0 and q_1=q_2=0.5, the same values used in [6]. State trajectories and input histories (Figures 1.3 and 1.4) for the inclusion of linear only, and linear and cubic only, terms in J_{ss} appear nearly identical to those in Carrington's example. The advantage of the Dwyer, Sena approach is the closed-form evaluation of the integral expression for $J_{ss}^{(n)}$ compared to the solution of the coupled differential equations in the Carrington approach.

Neither method, however, is without its difficulties. The amount of computation for a fairly simple spacecraft maneuver is significant. When multi-axis control is desired, computation increases substantially. Though either algorithm shows rapid convergence as higher order terms are included, neither approach gives an exact answer to the optimization problem. If a specified (finite) final time is desired, solution of the Ricatti differential equation is required, which is still feasible in closed form by the method of Juang, Turner and Chun [21]. The higher order terms are still able to be found in closed form.

A more appropriate choice of orientation parameter for the single-axis maneuver is the angular displacement, θ. Under the assumptions as outlined for this problem, the model would be linear

$$\Phi = \begin{bmatrix} 0 & \frac{1}{2} \\ 0 & 0 \end{bmatrix}$$

and the $\phi^{(n)}$ are n-homogeneous nonlinear terms. The solution can then be expressed as

$$\tau(t) = \psi^T \nabla J_{ss}(\xi(t))$$

where

$$J_{ss}(\xi) = \frac{1}{2} \xi^T P_{ss} \xi + \sum_{n=3}^{\infty} J_{ss}^{(n)}(\xi)$$

$$0 = P_{ss}\Phi + \Phi^T P_{ss} \psi \psi^T P_{ss} + Q$$

$$J_{ss}^{(n)}(\xi) = \int_0^{\infty} K_{ss}^{(n)}(e^{\tilde{\Phi}t}\xi)dt$$

$$K_{ss}^{(n)}(\xi) = \sum_{k=2}^{n-1} (\nabla J_{ss}^{(k)}(\xi))^T \phi^{(n+1-k)}(\xi)$$

$$- \frac{1}{2} \sum_{k=3}^{n-1} (\nabla J_{ss}^{(k)}(\xi))^T \psi \psi^T \nabla J_{ss}^{(n+2-k)}(\xi)$$

$$\tilde{\Phi} = \Phi - \psi \psi^T P_{ss}$$

and ∇ denotes the gradient with respect to ξ. For the problem stated here, $\tilde{\Phi}$ is a constant matrix and $e^{\tilde{\Phi}t}$ is easily computed.

The nature of the expansion of $\phi(\xi)$ results in terms appearing in $\tau(t)$ that are of odd order in the state variables. Thus, when only the first non-zero term in J_{ss} is used,

$$\tau = \delta_2 \gamma + \delta_2 \omega$$

and when the second non-zero term is included

Although present technology requires external thrusters to be operated in an on/off mode only, variable-thrust gas jets are currently under development [20]. Following the lead of Junkins, Turner, and Carrington [4,6], continuous torque maneuvers are designed for both internal momentum wheels and external thrusters.

1.3 A Direct Solution Approach

To give an appreciation of the amount and complexity of work involved in solving the nonlinear problem directly, an approach suggested by Dwyer and Sena [18] is here applied to the single-axis model (12a) and (12b). Parameters are chosen to allow for straightforward comparison to a similar example in [6]. The problem can be stated in a format compatible with Dwyer and Sena as

$$\text{minimize } J = \frac{1}{2}\int_0^\infty (\xi^T Q \xi + \tau^2)dt$$

$$\text{subject to } \dot{\xi} = \phi(\xi) + \psi\,\tau$$

$$\xi(o) = (0.7071,\ 0.0)^T$$

where $\xi = (\gamma,\omega)^T$, $\psi = (0, \frac{1}{I_o})^T$, $Q = \text{diag}\ (q_1, q_2)$, and

$$\phi(\xi) = \begin{bmatrix} \frac{1}{2}\ \sqrt{1-\gamma^2}\ \omega \\ 0 \end{bmatrix}.$$

Expansion of $\phi\ (\xi)$ results in

$$\phi\ (\xi) = \Phi\,\xi + \sum_{n=3}^{\infty} \phi^{(n)}$$

where

$$\underline{h}(t) = \underline{h}(\underline{\gamma}(t)) = C(\underline{\gamma}(t))C(\underline{\gamma}(t_o))^{-1}\underline{h}(t_o)$$

where $C(\underline{\gamma})$ is the direction cosine matrix written in terms of the Euler parameters and the vector product operation x such that

$$C(\underline{\gamma})\underline{v} = (2\gamma_o^2-1)\underline{v} + 2\underline{\gamma}\underline{\gamma}^T\underline{v} - 2\gamma_o\underline{\gamma} \times \underline{v}.$$

Under the condition $||\underline{\gamma}(t)|| < 1$ for all t, the model of spacecraft attitude control with internal momentum wheels then becomes

$$\dot{\underline{\gamma}} = \Gamma(\underline{\gamma})\underline{\omega} \tag{13a}$$

$$\dot{\underline{\omega}} = (I_o-I_a)^{-1}\ (\underline{h}(\underline{\gamma}) \times \underline{\omega}) - (I_o-I_a)^{-1}\ \underline{\mu}. \tag{13b}$$

Further simplification occurs when a rest-to-rest maneuver is to be controlled by internal torques and the initial wheel angular velocity is zero. Then $\underline{\omega}(t_o) = \underline{0}$, $\underline{\Omega}(t_o) = \underline{0}$; therefore

$$\underline{h}(t_o) = I_o\underline{\omega}(t_o) + I_a\underline{\Omega}(t_o) = \underline{0}$$

and (13b) reduces to

$$\dot{\underline{\omega}} = -\ (I_o-I_a)^{-1}\ \underline{\mu}.$$

Throughout the rest of this study, control will be assumed to be either external thrusters or internal momentum wheels but not both. Equations (11a) and (11b) or (13a) and (13b) are the respective mathematical models implied. For ease of reference, a general nonlinear model is defined

$$\dot{\underline{\xi}} = \underline{a}(\xi) + \mathcal{U}(\underline{\xi})\underline{\sigma}$$

where $\underline{\xi}^T = (\underline{\xi}_1^T,\underline{\xi}_2^T) = (\underline{\gamma}^T,\underline{\omega}^T)$ and $\underline{\sigma}$ is either $\underline{\tau}$ or $\underline{\mu}$ as dictated by the selected control.

When the spacecraft has a diagonal I_o matrix and when the control problem is a single-axis maneuver, further reduction of the model is possible. If rotation is desired about the i^{th} axis, ω_j and τ_j are set to zero for j=1,2,3, j≠i, and

$$\dot{\gamma}_i = \frac{1}{2}\gamma_o\omega_i$$
$$\dot{\omega}_i = \frac{1}{I_{oii}}\tau_i$$

With the earlier assumption on the norm of $\underline{\gamma}$, and dropping the subscript i,

$$\dot{\gamma} = \frac{1}{2}\sqrt{1-\gamma^2}\,\omega \tag{12a}$$

$$\dot{\omega} = \frac{1}{I_o}\tau. \tag{12b}$$

This reduced model is included to permit later comparison of the results of Carrington and Junkins [6] with an approach by Dwyer and Sena [18]. Introduction of the linearizing transformations will also begin with this model.

If the spacecraft orientation is to be controlled by the momentum wheels alone, external torques are zero ($\underline{\tau} = \underline{0}$) and total angular momentum in an inertial frame is constant. Junkins, Vadali, and Carrington [5,6] found that by judicious choice of another reference frame, this assumption allows the dynamics equations to be simplified and the order of the model to be reduced. Dwyer [19] showed that given the initial value, angular momentum in body coordinates could be propagated as a function of the Euler parameters $\gamma_o,\underline{\gamma}$ by

$$\dot{\underline{\gamma}} = \Gamma(\underline{\gamma})\underline{\omega} \tag{8a}$$

$$(I_o - I_a)\dot{\underline{\omega}} = \underline{h} \times \underline{\omega} + \underline{\tau} - \underline{\mu} \tag{8b}$$

$$(I_o - I_a)\dot{\underline{\Omega}} = \underline{\omega} \times \underline{h} - \underline{\tau} + I_o I_a^{-1}\underline{\mu} \tag{8c}$$

Several simplifications can be made for specific cases. If the momentum wheels are locked with respect to the spacecraft and control is to be by external thrusters only, then $\underline{\Omega} = \underline{0}$, $\dot{\underline{\Omega}} = \underline{0}$, $\underline{h} = I_o\underline{\omega}$ and (8b) and (8c) become

$$(I_o - I_a)\dot{\underline{\omega}} = I_o\underline{\omega} \times \underline{\omega} + \underline{\tau} - \underline{\mu} \tag{9}$$

$$\underline{0} = \underline{\omega} \times I_o\underline{\omega} - \underline{\tau} + I_o I_a^{-1}\,\underline{\mu}. \tag{10}$$

Control by external torques alone does not imply $\underline{\mu} = 0$. Indeed, from (10)

$$\underline{\mu} = I_a I_o^{-1}\,(I_o\underline{\omega} \times \underline{\omega} + \underline{\tau}).$$

Inserting this expression in (9) and simplifying gives

$$I_o\dot{\underline{\omega}} = I_o\underline{\omega} \times \underline{\omega} + \underline{\tau}.$$

Then under the condition $||\underline{\gamma}(t)|| < 1$ for all t, the model of spacecraft attitude control with external jets becomes

$$\dot{\underline{\gamma}} = \Gamma(\underline{\gamma})\underline{\omega} \tag{11a}$$

$$\dot{\underline{\omega}} = I_o^{-1}\,(I_o\underline{\omega} \times \underline{\omega}) + I_o^{-1}\,\underline{\tau} \tag{11b}$$

where I_o is the matrix of moments and products of inertia for the complete spacecraft system about its principal axes, I_a is a diagonal matrix of moments of inertia of the wheels about their respective axes (presuming only three orthogonal wheels are active), and $\underline{\Omega}$ is the vector of wheel angular velocities with respect to the body frame. Inserting (3) into (2) and dropping the subscript on $\underline{h}$,

$$I_o\dot{\underline{\omega}} + I_a\dot{\underline{\Omega}} + \underline{\omega} \times \underline{h} = \underline{\tau}. \tag{4}$$

The motor torque, $\underline{\mu}$, required to change wheel angular velocity $\underline{\Omega}$ (and through conservation also change spacecraft angular velocity $\underline{\omega}$) is given by

$$\underline{\mu} = I_a(\dot{\underline{\omega}} + \dot{\underline{\Omega}}). \tag{5}$$

Solving for $I_a\dot{\underline{\Omega}}$ and substituting in (4) gives the equations of evolution for the spacecraft angular velocities,

$$(I_o - I_a)\dot{\underline{\omega}} = \underline{h} \times \underline{\omega} + \underline{\tau} - \underline{\mu}. \tag{6}$$

Similarly, solving (5) for $\dot{\underline{\omega}}$ and substituting in (4) yields the wheel dynamics,

$$(I_o - I_a)\dot{\underline{\Omega}} = \underline{\omega} \times \underline{h} - \underline{\tau} + I_o I_a^{-1} \underline{\mu}. \tag{7}$$

The development of the dynamics equations given here parallels that in [17].

The system model is then given by (1) for the kinematics and (6) and (7) for the dynamics, repeated here for completeness.

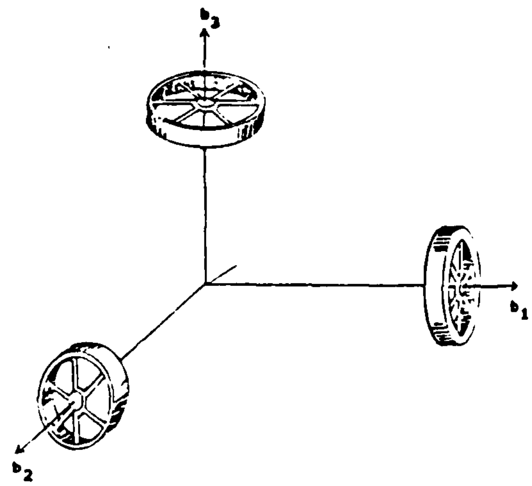


Figure 1.1. Three reaction wheel spacecraft attitude control system.

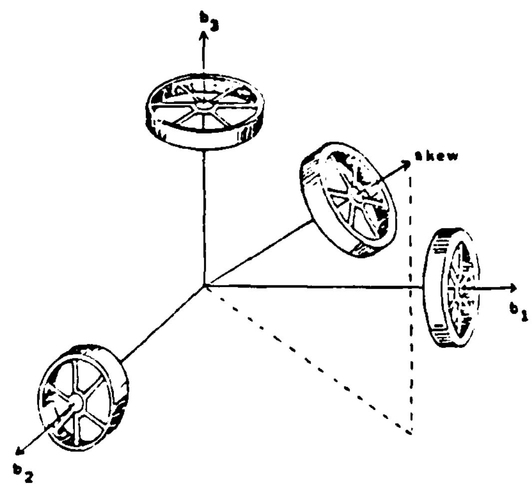


Figure 1.2. Four reaction wheel spacecraft attitude control system.

$$\Gamma(\underline{\gamma})\underline{\omega} = \frac{1}{2}\,(\gamma_o\underline{\omega} + \underline{\gamma} \times \underline{\omega})$$

or as a matrix

$$\Gamma(\underline{\gamma}) = \frac{1}{2}\begin{bmatrix} \gamma_o & -\gamma_3 & \gamma_2 \\ \gamma_3 & \gamma_o & -\gamma_1 \\ -\gamma_2 & \gamma_1 & \gamma_o \end{bmatrix}.$$

Spacecraft attitude maneuvers are effected by the application of external torques (symmetric pairs of jets, or thrusters) or internal torques (momentum transfer wheels, or simply momentum wheels). When three wheels are used, they are typically aligned so their axes of rotation coincide with the principal axes of the spacecraft (Fig. 1.1). If a fourth wheel is included, it is skewed with respect to the other three (Fig. 1.2).

Dynamics equations for a rotational maneuver result from application of the law of conservation of angular momentum. The rate of change of angular momentum in an inertial frame is equal to the external torque applied. Because torques are applied along the spacecraft axes, angular momentum is referenced to the body frame, but the rotation of the frame must then be accounted for [16],

$$\dot{\underline{h}}_{inertial} = \dot{\underline{h}}_{body} + \underline{\omega} \times \underline{h}_{body} = \underline{\tau} \qquad (2)$$

where $\underline{h}$ is the angular momentum vector referenced as shown by the subscripts, and $\underline{\tau}$ represents external torques supplied by thrusters. The total angular momentum of the spacecraft is given by

$$\underline{h}_{body} = I_o\underline{\omega} + I_a\underline{\Omega} \qquad (3)$$

$$\beta_o^2 + \beta_1^2 + \beta_2^2 + \beta_3^2 = \cos^2(\tfrac{1}{2}\theta) + n_1^2\sin^2(\tfrac{1}{2}\theta) + n_2^2\sin^2(\tfrac{1}{2}\theta) + n_3^2\sin^2(\tfrac{1}{2}\theta)$$
$$= \cos^2(\tfrac{1}{2}\theta) + (n_1^2 + n_2^2 + n_3^2)\sin^2(\tfrac{1}{2}\theta)$$
$$= 1.$$

Therefore β_o is related to $\underline{\beta}'$ by the expression

$$\beta_o = \pm\sqrt{1 - \underline{\beta}'^T\underline{\beta}'}.$$

To resolve the redundancy inherent in the full Euler parameter set, a reduced set $\underline{\gamma}$ is defined by $\underline{\gamma} = \underline{\beta}'$ if $\beta_o > 0$, or $\underline{\gamma} = -\underline{\beta}'$ if $\beta_o < 0$, and $\gamma_o = |\beta_o|$. An implied constraint in this definition is $||\underline{\gamma}|| \leq 1$ for any maneuver.

As the spacecraft orientation changes, so do the angle θ and the unit vector $\underline{n}$. Whittaker [15] shows that the components of angular velocity with respect to the body frame can be related to the Euler parameters and their time derivatives. Algebraic manipulation yields the expression for the time rate of change of the parameters,

$$\begin{aligned}
\dot{\gamma}_o &= \frac{1}{2}(-\gamma_1\omega_1 - \gamma_2\omega_2 - \gamma_3\omega_3) \\
\dot{\gamma}_1 &= \frac{1}{2}(\gamma_o\omega_1 - \gamma_3\omega_2 + \gamma_2\omega_3) \\
\dot{\gamma}_2 &= \frac{1}{2}(\gamma_3\omega_1 + \gamma_o\omega_2 - \gamma_1\omega_3) \\
\dot{\gamma}_3 &= \frac{1}{2}(-\gamma_2\omega_1 + \gamma_1\omega_2 + \gamma_o\omega_3)
\end{aligned}$$

where $\underline{\omega} = (\omega_1, \omega_2, \omega_3)^T$ is the spacecraft angular velocity vector. The reduced kinematic model may then be written as

$$\dot{\underline{\gamma}} = \Gamma(\underline{\gamma})\underline{\omega} \tag{1}$$

where $\Gamma(\underline{\gamma})$ can be considered an operator such that

in such a way as to allow specification of the characteristics of the resulting linear equivalent system. The application of interest to Freund was 3- and 4-degree of freedom robotic manipulators.

The purpose of the work described in this study was to investigate application of the theory of linearizing transformations to the nonlinear model of spacecraft attitude control. When a linear equivalent system had been defined, strategies for dealing with variations in model parameters and constraints on control variables were studied.

The remainder of Chapter One is given to development of the spacecraft attitude control model and one direct solution approach. Three algorithms for determining a linear equivalent system are described in Chapter Two. Formulation and solution of optimization problems for the linear system are discussed, and simulations are presented in Chapter Three. The fourth chapter deals with parameter variations, and a singularity problem that arises due to the use of a reduced set of Euler parameters. Compensation for commands in excess of input constraints is treated in Chapter Five. Recommendations for further study are presented in conclusion.

1.2. The Spacecraft Attitude Control Model

An arbitrary orientation of a spacecraft with respect to an inertial frame can be represented as a single, virtual rotation about a unit vector $\underline{n}$ which is fixed relative to both the spacecraft, or body, frame and the inertial frame [14]. This representation is alternatively described by a set of Euler parameters $\underline{\beta} = (\beta_o, \beta_1, \beta_2, \beta_3)^T = (\beta_o, \underline{\beta}'^T)^T$ where $\beta_o = \cos(\frac{1}{2}\theta)$ and $\beta_i = n_i \sin(\frac{1}{2}\theta)$, $i = 1,2,3$. The Euler parameters are not an independent set, for from the definition,

Carrington and Junkins [6] assumed a feedback control that was a polynomial, rather than linear, function of the state. This approach requires the solution of a matrix Ricatti equation for the coefficients of the linear feedback terms. Higher order coefficients are found by solving a coupled set of linear nonhomogeneous differential equations which depend on the lower order coefficients. Theory was presented to include either fixed or free final time, but examples only illustrated the steady-state case. The number of terms in the nonlinear feedback required for satisfactory convergence is apparently a function of the desired maneuver. For the examples presented, third order polynomials in the state appeared to be sufficient.

Each of the approaches mentioned solves the nonlinear problem directly. Each depends on convergence to a satisfactory trajectory of an iterative process or a polynomial series. The convergence appears to be well-behaved for the examples presented, but the result must be considered to be approximate.

Other authors have sought to solve nonlinear control problems indirectly, by in some manner deriving a "linear equivalent" system. Improving on the early work of Brockett [7], Hunt, Su and Meyer [8,9] proposed the derivation of a nonlinear change of variables which would transform the nonlinear system to a linear system in Brunovsky canonical form. Sufficient conditions were determined for assurance of the existence of such a transformation. They extended the theory to multi-input systems and presented examples of applications to flight control systems [10,11].

Freund [12,13] developed an algorithm for decoupling nonlinear systems. In the process of decoupling, free parameters were introduced

$$\dot{\theta} = \omega$$

$$\dot{\omega} = \frac{1}{I_o} \tau$$

and the control easily determined. The point of presenting the Carrington and Dwyer approaches in the context of the single-axis maneuver is to demonstrate the degree of difficulty involved in computing the solution to the simplest required maneuver when the equations of motion are nonlinear.

The desire then is to find a way of formulating the multi-axis spacecraft attitude control problem that gives exact results for either specified or unspecified final time and which are computationally easier to implement than those described above. This motivation provides the rationale for investigating the application of linearizing transformations.

CHAPTER TWO

LINEARIZING TRANSFORMATIONS

The objective of linearizing transformation theory is to find a change of variables such that a nonlinear system is described in the transformed space as an equivalent linear system. This transformation will be nonlinear itself, and the equivalence may be valid over the whole or only a portion of the operating universe. A generalized problem can be stated

$$\text{given} \qquad \dot{\underline{\xi}} = \underline{\phi}(\underline{\xi}) + \Psi(\underline{\xi})\underline{\sigma} \tag{14}$$

find X and U

such that $\underline{x} = X(\underline{\xi})$, $\underline{u} = U(\underline{\xi},\underline{\sigma})$

$$\text{and} \qquad \dot{\underline{x}} = A\underline{x} + B\underline{u} \tag{15}$$

for some state- and input-independent matrices A and B. (For the present effort, A and B are considered constant.) If such a transform pair X,U can be constructed, then the control problem can be formulated and solved for the linear ($\underline{x}$,$\underline{u}$) system using the powerful known linear techniques. The input $\underline{u}$ is then transformed to nonlinear ($\underline{\xi}$,$\underline{\sigma}$) system space, and the maneuver completed. The input to the nonlinear

system is found by $\underline{\sigma} = U^{-1}(\underline{\xi},\underline{u}) = W(\underline{x},\underline{u})$ where the inverse exists. Usefulness of this approach is eventually determined by the region over which U^{-1} is well defined.

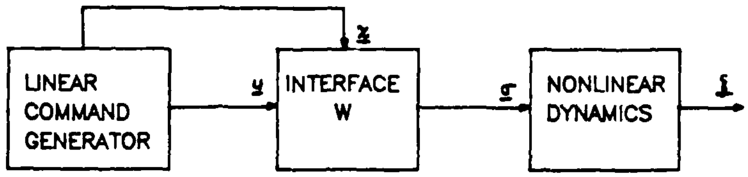
As in linear systems theory, control can be implemented in either open-loop or closed-loop form as shown in Fig. 2.1. The linear problem can be solved and analytic expressions for $\underline{x}(t)$ and $\underline{u}(t)$ obtained. The command generator supplies not only the linear input histories, but also the linear state trajectories. Both are fed to the input transformation W to produce the nonlinear input $\underline{\sigma}$ which drives the system (Fig. 2.1(a)). If use of the nonlinear state trajectory in determining $\underline{\sigma}$ is preferred, $U^{-1}(\underline{\xi},\underline{u})$ can be used in place of $W(\underline{x},\underline{u})$ (Fig. 2.1(b)).

Closed loop control is effected by transforming the state variable to linear space and forming the linear feedback control $\underline{u} = \underline{u}(\underline{x})$. The control $\underline{u}$ is supplied to the transformation W along with the linear state vector (Fig. 2.1(c)). As before, $\underline{\xi}$ can be used instead of $\underline{x}$ by replacing W by U^{-1}.

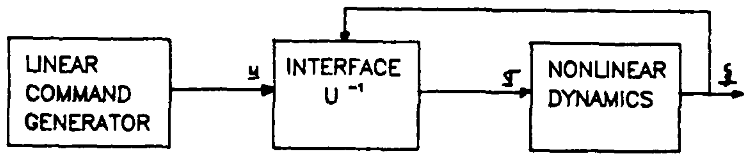
Three approaches to finding an equivalent linear system will be discussed.

2.1 Hunt, Su, Meyer Technique

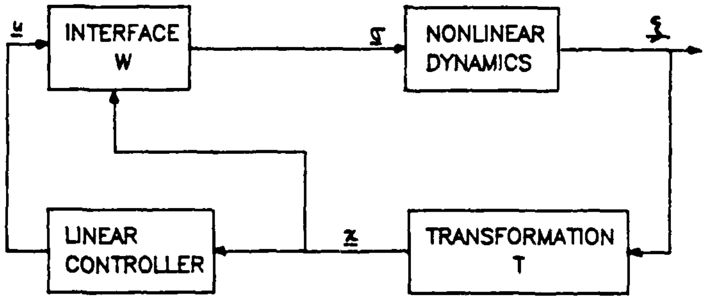
In their paper "Global Transformations of Nonlinear Systems" [9], Hunt, Su and Meyer presented sufficient conditions for existence of a globally equivalent linear system. Their technique presumes a nonlinear system of the form (14) with a scalar input σ, thus the matrix $\mathcal{W}(\underline{\xi})$ reduces to a vector field $\underline{\Psi}(\underline{\xi})$. A method for constructing the transformation between nonlinear and equivalent linear systems was introduced. This algorithm is dependent upon the solution, by the method of characteristics, of a system of linear partial differential



(a) Open loop control, interface driven by linear model state.



(b) Open loop control, interface driven by nonlinear model state.



(c) Closed loop control, interface driven by linear model state.

Figure 2.1. Spacecraft attitude control using an equivalent linear system for input determination.

equations. When the sufficient conditions are satisfied and the system of differential equations can be solved, the resulting linear system for the single input case is in the form of a series of integrators,

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ & & \cdot & & \\ & & \cdot & & \\ & & \cdot & & \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{bmatrix}.$$

A few definitions are required.

Let $\phi(\xi)$ and $\Psi(\xi)$ be infinitely differentiable vector fields on R^n. The Lie bracket of Ψ with respect to ϕ is also a vector field on R^n, and is defined by

$$[\phi, \Psi] = \frac{d\Psi}{d\xi}\phi - \frac{d\phi}{d\xi}\Psi .$$

Further, define

$$(ad^0\ \phi, \Psi) = \Psi$$

$$(ad^1\ \phi, \Psi) = [\phi, \Psi]$$

$$\vdots$$

$$(ad^k\ \phi, \Psi) = [\phi, (ad^{k-1}\ \phi, \Psi)].$$

Finally, let X_i be a function on R^n which is infinitely differentiable. Then the Lie derivative of X_i with respect to ϕ is defined

$$\langle dX_i, \phi\rangle = \frac{dX_i}{d\xi_1}\phi_1 + \frac{dX_i}{d\xi_2}\phi_2 + \cdots + \frac{dX_i}{d\xi_n}\phi_n .$$

The algorithm for a single input is summarized as follows:

1. Compute $(ad^k \underline{\phi}, \underline{\psi})$ for $k=0,1,\ldots,n-1$.

2. Construct the initial curves by solving

$$\frac{d\underline{\xi}}{dt_1} = (ad^{n-1} \underline{\phi}, \underline{\psi}), \quad \underline{\xi}(0) = \underline{0}$$

to get $\underline{\xi}(t_1)$. (n equations)

3. Build the integral surface by successively solving

$$\frac{d\underline{\xi}}{dt_k} = (ad^{n-k} \underline{\phi}, \underline{\psi}), \quad \underline{\xi}(t_1,\ldots,t_{k-1},0) = \underline{\xi}(t_1,\ldots,t_{k-1})$$

for $k=2,3,\ldots,n$. (n equations for each k)

4. The solutions to the partial differential equation give $\underline{\xi}$ as a function of $(t_1,t_2,\ldots,t_n)$. Solve for t_k, $k=1,2,\ldots,n$ as a function of $(\xi_1,\xi_2,\ldots,\xi_n)$.

5. Set $X_1=t_1(\xi_1,\xi_2,\ldots,\xi_n)$ and solve for successive X_k's by

$$X_k = \langle dX_{k-1}, \underline{\phi}\rangle.$$

6. Find U by

$$U = \langle dX_n, \underline{\phi} + \underline{\psi}\,\sigma\rangle.$$

Application of this algorithm to the single-axis spacecraft attitude control problem (12a and 12b) is straightforward and clarifies the procedure. Define $\underline{\xi} = (\xi_1,\xi_2)^T = (\gamma,\omega)^T$ and $\sigma = \tau$. Then $\underline{\phi}$ and $\underline{\psi}$ are given as

$$\underline{\phi} = \begin{bmatrix} \frac{1}{2}\sqrt{1-\xi_1^2}\,\xi_2 \\ 0 \end{bmatrix}, \quad \underline{\psi} = \begin{bmatrix} 0 \\ \frac{1}{I_o} \end{bmatrix}.$$

Computing the necessary Lie derivatives,

$$(ad^0\, \underline{g}, \underline{\psi}) = \begin{bmatrix} 0 \\ \frac{1}{I_o} \end{bmatrix}$$

$$(ad^1\, \underline{g}, \underline{\psi}) = [\underline{g}, \underline{\psi}] = \begin{bmatrix} -\frac{1}{2I_o}\sqrt{1-\xi_1^2} \\ 0 \end{bmatrix}.$$

The system of partial differential equations

$$\frac{d\xi_1}{dt_1} = -\frac{1}{2I_o}\sqrt{1-\xi_1^2}\,, \quad \xi_1(0) = 0$$

$$\frac{d\xi_1}{dt_2} = 0, \qquad \xi_1(t_1,0) = \xi_1(t_1)$$

$$\frac{d\xi_2}{dt_1} = 0, \qquad \xi_2(0) = 0$$

$$\frac{d\xi_2}{dt_2} = \frac{1}{I_o}, \qquad \xi_2(t_1,0) = \xi_2(t_1)$$

is solved by $\xi_1(t_1,t_2) = -\sin(\frac{1}{2I_o}t_1)$ and $\xi_2(t_1,t_2) = \frac{1}{I_o}t_2$. These are easily inverted for t_1 and t_2 as $t_1(\xi_1,\xi_2) = -2I_o\sin^{-1}(\xi_1)$, $t_2(\xi_1,\xi_2) = I_o\xi_2$. Then setting $X_1(\underline{\xi}) = t_1 = -2I_o\sin^{-1}(\xi_1)$, X_2 and U are found,

$$X_2 = \langle dX_1, \underline{g}\rangle = -I_o\xi_2$$

$$U = \langle dX_2, \underline{f} + \underline{g}\sigma \rangle = -\sigma.$$

The negative signs that appear throughout may be dropped and the resulting transformation in terms of the spacecraft parameters is defined

$$x_1 = 2I_o \sin^{-1}\gamma$$

$$x_2 = I_o\omega$$

$$u = \tau.$$

To verify that the desired linear system has been achieved, differentiate x_1 and x_2 with respect to time using the chain rule.

$$\dot{x}_1 = 2I_o \frac{1}{\sqrt{1 - \gamma^2}} \dot{\gamma} = I_o\omega = x_2$$

$$\dot{x}_2 = I_o\dot{\omega} = \tau = u.$$

Therefore

$$\dot{\underline{x}} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \underline{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u.$$

As mentioned in Chapter One, the single axis control problem is very quickly solved by the use of angular displacement and angular velocity for the state variables. The linear equivalent system was developed here to introduce the use of the Hunt, Su, Meyer technique in an easily understood context.

A later publication [10] extends the algorithm outlined above to multi-input nonlinear systems. The objective linear equivalent system becomes a Brunovsky canonical system and is described by blocks of series of integrators, where the number of blocks and the number of

inputs are equal. Kronecker indices are introduced, specifying the number of integrators driven by each input. Multiple inputs add little complexity to the description of the algorithm. The input matrix $\mathcal{H}(\underline{\xi})$ is partitioned

$$\mathcal{H}(\underline{\xi}) = [\underline{\psi}_1(\underline{\xi}) \mid \underline{\psi}_2(\underline{\xi}) \mid \cdots \mid \underline{\psi}_m(\underline{\xi})]$$

where m is the number of inputs. Successive Lie brackets of $\underline{\psi}_i(\underline{\xi})$ with respect to $\underline{\phi}(\underline{\xi})$ are calculated for i=1,2,...,m, and a system of partial differential equations is defined as before. If this system of equations can be solved for $\underline{\xi}$ as a function of $(t_1,t_2,...,t_n)$, and if that function can be inverted, then the transformation is constructed from the characteristic parameters associated with the highest order Lie bracket of each $\underline{\psi}_i(\underline{\xi})$.

Consider now the model presented in (11a) and (11b), and assume a matrix $I_o = \mathrm{diag}(I_1,I_2,I_3)$. Formatting for application of this algorithm gives $\underline{\xi} = (\underline{\xi}_1^T,\underline{\xi}_2^T)^T = (\underline{\gamma}^T,\underline{\omega}^T)^T$, $\underline{\sigma} = \underline{\tau}$, and

$$\underline{\phi}(\underline{\xi}) = \begin{bmatrix} \Gamma(\underline{\xi}_1)\underline{\xi}_2 \\ \underline{0} \end{bmatrix} \qquad \mathcal{H}(\underline{\xi}) = \begin{bmatrix} 0 \\ I_o^{-1} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1/I_1 & 0 & 0 \\ 0 & 1/I_2 & 0 \\ 0 & 0 & 1/I_3 \end{bmatrix}.$$

The required Lie brackets are calculated (see Table 2.1), and the system of partial differential equations defined.

Table 2.1. Lie Derivatives for the Multi-axis Control Problem

$$(\mathrm{ad}^{0}\underline{g},\underline{\psi}_1) = \underline{\psi}_1 = (0,\ 0,\ 0,\ 1/I_1,\ 0,\ 0)^T$$

$$(\mathrm{ad}^{0}\underline{g},\underline{\psi}_2) = \underline{\psi}_2 = (0,\ 0,\ 0,\ 0,\ 1/I_2,\ 0)^T$$

$$(\mathrm{ad}^{0}\underline{g},\underline{\psi}_3) = \underline{\psi}_3 = (0,\ 0,\ 0,\ 0,\ 0,\ 1/I_3)^T$$

$$(\mathrm{ad}^{1}\underline{g},\underline{\psi}_1) = [\underline{g},\underline{\psi}_1] =$$

$$\left(-\frac{1}{2I_1}\sqrt{1-||\underline{\xi}_1||^2},\ -\frac{1}{2I_1}\xi_{13},\ \frac{1}{2I_1}\xi_{12},\ 0\ \frac{I_1-I_3}{I_1I_2}\xi_{23},\ \frac{I_2-I_1}{I_1I_3}\xi_{22}\right)^T$$

$$(\mathrm{ad}'\underline{g},\underline{\psi}_2) = [\underline{g},\underline{\psi}_2] =$$

$$\left(\frac{1}{2I_2}\xi_{13},\ -\frac{1}{2I_2}\sqrt{1-||\underline{\xi}_1||^2},\ -\frac{1}{2I_2}\xi_{11},\ \frac{I_3-I_2}{I_1I_2}\xi_{23},\ 0\ \frac{I_2-I_1}{I_2I_3}\xi_{21}\right)^T$$

$$(\mathrm{ad}'\underline{g},\underline{\psi}_3) = [\underline{g},\underline{\psi}_3] =$$

$$\left(-\frac{1}{2I_3}\xi_{12},\ \frac{1}{2I_3}\xi_{11},\ -\frac{1}{2I_3}\sqrt{1-||\underline{\xi}_1||^2},\ \frac{I_3-I_2}{I_1I_3}\xi_{22},\ \frac{I_1-I_2}{I_2I_3}\xi_{21},\ 0\right)^T$$

$$\frac{d\underline{\xi}}{dt_1} = [\underline{g}, \underline{\psi}_1] \ , \ \underline{\xi}(0) = \underline{0}$$

$$\frac{d\underline{\xi}}{dt_2} = [\underline{g}, \underline{\psi}_2] \ , \ \underline{\xi}(t_1, 0) = \underline{\xi}(t_1)$$

$$\frac{d\underline{\xi}}{dt_3} = [\underline{g}, \underline{\psi}_3] \ , \ \underline{\xi}(t_1, t_2, 0) = \underline{\xi}(t_1, t_2)$$

$$\frac{d\underline{\xi}}{dt_4} = \underline{\psi}_1 \ , \ \underline{\xi}(t_1, t_2, t_3, 0) = \underline{\xi}(t_1, t_2, t_3)$$

$$\frac{d\underline{\xi}}{dt_5} = \underline{\psi}_2 \ , \ \underline{\xi}(t_1, t_2, t_3, t_4, 0) = \underline{\xi}(t_1, t_2, t_3, t_4)$$

$$\frac{d\underline{\xi}}{dt_6} = \underline{\psi}_3 \ , \ \underline{\xi}(t_1, t_2, t_3, t_4, t_5, 0) = \underline{\xi}(t_1, t_2, t_3, t_4, t_5)$$

Because of the structure of $\underline{g}(\underline{\xi})$, $\Psi(\underline{\xi})$, and their Lie derivatives for this problem, the parameters t_1, t_2, and t_3 will appear in solutions for $\underline{\xi}_1$ and not in solutions for $\underline{\xi}_2$. Similarly, t_5, t_6 and t_7 will appear in solutions for $\underline{\xi}_2$ alone. Since the transformation is to be constructed from the parameters associated with the highest order Lie derivatives, only the three equations involving $\underline{\xi}_1$ in each of the first three sets of equations need to be solved. (Note that in general, for a system of order n, a total of n^2 partial differential equations must be solved and that solution set inverted.) If the nine specified equations can be solved and analytical expressions found for t_1, t_2, and t_3 in terms of $\underline{\xi}_1$, the algorithm can proceed and the transformation constructed. To this point, that problem is unsolved. However, the attempt at applying the Hunt, Su, Meyer algorithm to spacecraft attitude control problems is not unfruitful. A particular strategy for choosing

the linear state space variables is suggested by the integrator structure of the desired equivalent linear system.

2.2 Block Triangular Case

Combining the objectives that the state transformation should be input independent ($X = X(\underline{\xi})$), the input transformation should depend explicitly on $\underline{\sigma}$ ($U = U(\underline{\xi},\underline{\sigma})$), and the linear system should be a series of integrators, Dwyer [19,22] adapted a technique of Meyer's applicable to nonlinear systems in block triangular form. The technique can be nonrigorously described as follows:

Given the nonlinear system (14),

1. Find a set of elements of the state vector $\underline{\xi}$ such that
   (a) the total of the selected elements and as many time derivatives as are independent of the inputs is equal to the order of the system, and
   (b) all inputs are introduced upon the next time differentiation of the selected elements.
2. Define X to be the set of elements and derivatives found in step 1(a).
3. Define U to be the set of subsequent derivatives found in step 1(b).

The single-axis control problem again serves to illustrate the algorithm. For the model

$$\dot{\gamma} = \frac{1}{2} \sqrt{1-\gamma^2}\ \omega$$
$$\dot{\omega} = \frac{1}{I_o} \tau$$

n=2 and two elements and/or derivatives must be selected. Since γ and $\dot{\gamma}$ are both independent of the input, and

$$\ddot{\gamma} = \frac{d\dot{\gamma}}{d\gamma}\,\dot{\gamma} + \frac{d\dot{\gamma}}{d\omega}\,\dot{\omega} = -\frac{1}{4}\gamma\omega^2 + \frac{1}{2I_o}\sqrt{1-\gamma^2}\,\tau$$

explicitly introduces the one input τ, that choice is made. Then define $\underline{x}$ as

$$x_1 = X_1(\gamma,\omega) = \gamma$$

$$x_2 = X_2(\gamma,\omega) = \dot{\gamma} = \frac{1}{2}\sqrt{1-\gamma^2}\,\omega$$

and u is given by

$$u = U(\gamma,\omega,\tau) = \ddot{\gamma} = -\frac{1}{4}\gamma\omega^2 + \frac{1}{2I_o}\sqrt{1-\gamma^2}\,\tau.$$

The resulting linear system is in Brunovsky canonical form

$$\dot{\underline{x}} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \underline{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u.$$

Also important to note is that a requirement to drive the nonlinear system to the origin $(\gamma,\omega) = (0,0)$ is mirrored in the linear model by a requirement to drive (x_1,x_2) to $(0,0)$. When the problem is solved in terms of the linear system, nonlinear input commands are obtained by inverting $U(\gamma,\omega,\tau)$,

$$\tau = U^{-1}(\gamma,\omega,u) = \frac{2I_o}{\sqrt{1-\gamma^2}}\left(u + \frac{1}{4}\gamma\omega^2\right).$$

Knowing the desired trajectory for $\underline{x}(t)$, the transformation X can also be inverted to find desired trajectories for γ and ω,

$$\gamma = x_1$$

$$\omega = \frac{2}{\sqrt{1-\gamma^2}}\,\dot{\gamma} = \frac{2}{\sqrt{1-\gamma^2}}\,x_2.$$

A singularity occurs in the expressions for τ and ω when $\gamma=1$. But the assumption was made in developing the model in Chapter One that $\|\underline{\chi}\| < 1$. For the single-axis problem, that condition translates to $-1 < \gamma < 1$, and under this condition the singularity will not be excited.

This method of finding a linear equivalent system was applied to the multi-axis spacecraft maneuver controlled by internal torques [19], and external torques [22]. In both cases, choice of the linear state variables follows from the single-axis problem above, and the desired result is achieved.

For the external thrusters model (11a) and (11b),

$$\underline{x}_1 = X_1(\underline{\chi},\underline{\omega}) = \underline{\chi}$$

$$\underline{x}_2 = X_2(\underline{\chi},\underline{\omega}) = \dot{\underline{\chi}} = \Gamma(\underline{\chi})\underline{\omega}$$

$$\underline{u}_1 = U(\underline{\chi},\underline{\omega},\underline{\tau}) = \ddot{\underline{\chi}} = \frac{d}{d\underline{\chi}}\,(\Gamma(\underline{\chi})\underline{\omega})\dot{\underline{\chi}} + \frac{d}{d\underline{\omega}}\,(\Gamma(\underline{\chi})\underline{\omega})\dot{\underline{\omega}}$$

$$= -\frac{1}{4}\underline{\chi}(\underline{\omega}^T\underline{\omega}) + \Gamma(\underline{\chi})I_o^{-1}\,(I_o\underline{\omega}\times\underline{\omega}) + \Gamma(\underline{\chi})I_o^{-1}\underline{\tau}.$$

(Linear acceleration associated with external torques will be designated $\underline{u}_1$, and that associated with internal torques will be designated $\underline{u}_2$ henceforth.) The state transformation can be inverted

$$\underline{\chi} = \underline{x}_1$$
$$\underline{\omega} = \Gamma^{-1}(\underline{\chi})\underline{x}_2$$

CHAPTER THREE

OPTIMIZATION PROBLEMS AND SIMULATIONS

3.1 The Performance Measure

To determine a desired system input presuming a target state normalized to zero, the modern control approach defines a measure of performance

$$J = J_p(\underline{\xi}(t_f)) + \int_{t_o}^{t_f} [J_q(\underline{\xi}(t),t) + J_r(\underline{\sigma}(t),t)]dt$$

which is then minimized subject to the state equations

$$\dot{\underline{\xi}} = \underline{\phi}(\underline{\xi}) + \Psi(\underline{\xi})\underline{\sigma}.$$

Typically J_p, J_q and J_r are chosen to be quadratic functions of the state and input. When the further condition of linear state equations is met, solution of the minimization problem reduces to the solution of a matrix Ricatti equation. In many cases this Ricatti equation can be solved and an optimum control strategy, in the sense of minimizing J, is obtained.

However, where the state equations are nonlinear the complexity of the solution is greatly increased. The nonlinear nature of the model for spacecraft attitude control (either 11(a) and 11(b) or 12(a) and 12(b)) results in the inability to obtain an exact solution to the

Free parameters in the generalized transformations allow use of a combination of classical and modern control theory to be applied. The open loop linear system poles can be placed as desired by appropriate choices of a_o and a_1. Linear inputs $\underline{u}$ can be determined by the solution of LQ (linear system, quadratic cost) optimization problems. Chapter Three presents several such optimization problems and results of associated simulations.

Table 2.3. Generalized Transformations for Spacecraft Attitude Control Using Internal Torques

Nonlinear Model

$$\dot{\underline{\gamma}} = \Gamma(\underline{\gamma})\underline{\omega}$$

$$\dot{\underline{\omega}} = (I_o - I_a)^{-1}(\underline{h}(\underline{\gamma}) x \underline{\omega}) - (I_o - I_a)^{-1}\underline{\mu}$$

Transformation

$$\underline{x}_1 = \underline{\gamma}$$

$$\underline{x}_2 = \dot{\underline{\gamma}} \qquad = \Gamma(\underline{\gamma})\underline{\omega}$$

$$\underline{u}_2 = \ddot{\underline{\gamma}} + a_1\dot{\underline{\gamma}} + a_o\underline{\gamma}$$

$$= -\frac{1}{4}\underline{\gamma}(\underline{\omega}^T\underline{\omega}) + \Gamma(\underline{\gamma})(I_o - I_a)^{-1}(\underline{h}(\underline{\gamma}) x \underline{\omega}) + a_1\Gamma(\underline{\gamma})\underline{\omega} + a_o\underline{\gamma} - \Gamma(\underline{\gamma})(I_o - I_a)^{-1}\underline{\mu}$$

Inverse Transformations

$$\underline{\gamma} = \underline{x}_1$$

$$\underline{\omega} = \Gamma^{-1}(\underline{\gamma})\underline{x}_2$$

$$\underline{\mu} = -\frac{1}{2\gamma_o}(\underline{\omega}^T\underline{\omega} - 4a_o)(I_o - I_a)\underline{\gamma} + \underline{h}(\underline{\gamma}) x \underline{\omega} + a_1(I_o - I_a)\underline{\omega} - (I_o - I_a)\Gamma^{-1}(\underline{\gamma})\underline{u}_2$$

Linear Model

$$\dot{\underline{x}}_1 = \underline{x}_2$$

$$\dot{\underline{x}}_2 = -a_o\underline{x}_1 - a_1\underline{x}_2 + \underline{u}_2$$

Table 2.2. Generalized Transformations for Spacecraft Attitude Control Using External Torques

Nonlinear Model

$$\dot{\underline{\gamma}} = \Gamma(\underline{\gamma})\underline{\omega}$$

$$\dot{\underline{\omega}} = I_o^{-1}(I_o\underline{\omega}x\underline{\omega})+I_o^{-1}\underline{\tau}$$

Transformation

$$\underline{x}_1 = \underline{\gamma}$$

$$\underline{x}_2 = \dot{\underline{\gamma}} \qquad = \Gamma(\underline{\gamma})\underline{\omega}$$

$$\underline{u}_1 = \ddot{\underline{\gamma}}+a_1\dot{\underline{\gamma}}+a_o\underline{\gamma} = -\tfrac{1}{4}\underline{\gamma}(\underline{\omega}^T\underline{\omega})+\Gamma(\underline{\gamma})I_o^{-1}(I_o\underline{\omega}x\underline{\omega})+a_1\Gamma(\underline{\gamma})\underline{\omega}+a_o\underline{\gamma}+\Gamma(\underline{\gamma})I_o^{-1}\underline{\tau}$$

Inverse Transformation

$$\underline{\gamma} = \underline{x}_1$$

$$\underline{\omega} = \Gamma^{-1}(\underline{\gamma})\underline{x}_2$$

$$\underline{\tau} = \frac{1}{2\gamma_o}(\underline{\omega}^T\underline{\omega}-4a_o)I_o\underline{\gamma}-I_o\underline{\omega}x\underline{\omega}-a_1I_o\underline{\omega}+I_o\Gamma^{-1}(\underline{\gamma})\underline{u}_1$$

Linear Model

$$\dot{\underline{x}}_1 = \underline{x}_2$$

$$\dot{\underline{x}}_2 = -a_o\underline{x}_1-a_1\underline{x}_2 + \underline{u}_1$$

$$+ a_1 \Gamma(\underline{\gamma})\underline{\omega} + a_o\underline{\gamma} - \Gamma(\underline{\gamma})(I_o - I_a)^{-1}\underline{\mu}.$$

The equivalent linear system is still the vector form of a damped second-order system with damping coefficients a_o and a_1 freely chosen.

2.4. Summary

Although the algorithm introduced by Hunt, Su and Meyer did not give an equivalent linear system for the multi-axis spacecraft attitude control problem, two benefits resulted from considering their work. First, the idea of even looking for a linear equivalent was introduced. Second, their objective of a linear system of series of integrators suggested a starting point for defining the transformations that eventually resulted. The two difficulties in their approach are solution of a set of n^2 partial differential equations and the inversion of that solution.

Dwyer's and Freund's techniques differ in two basic ways. Central in the Hunt-Su-Meyer and Dwyer approaches is the development of a transformation that will give an equivalent linear system. Freund's focus is on finding a nonlinear feedback control law which will decouple the system outputs. Secondly, the equivalent linear system resulting from the first two approaches is predestined to be a series of integrators, while the linear system described by the evolution of Freund's outputs is an arbitrarily damped second-order system.

Application of Dwyer's method to the spacecraft attitude control problem can be easily modified to achieve the generality provided by Freund. In place of $\underline{u} = \ddot{\underline{\gamma}}$, choose $\underline{u}$ to be $\underline{u} = \ddot{\underline{\gamma}} + a_1\dot{\underline{\gamma}} + a_o\underline{\gamma}$ instead. These generalized results are summarized in Tables 2.2 and 2.3 for easy reference.

$$\underline{\phi}(\underline{\xi}) = \begin{bmatrix} \Gamma(\underline{\gamma})\underline{\omega} \\ (I_o - I_a)^{-1}(\underline{h}(\underline{\gamma}) x \underline{\omega}) \end{bmatrix}, \quad \Psi = \begin{bmatrix} 0 \\ -(I_o - I_a)^{-1} \end{bmatrix}, \quad \underline{\delta} = \underline{\gamma}.$$

Calculation of $N_\phi^o(\underline{\gamma})$ and $N_\phi^1(\underline{\gamma})$ are the same as before. Differential orders d_1, d_2, d_3 are all still two, although for this case

$$\frac{d}{d\underline{\xi}}[N_\phi^1(\underline{\gamma})]\,\Psi = -\,\Gamma(\underline{\gamma})(I_o - I_a)^{-1}.$$

Developing Freund's nonlinear feedback

$$\underline{\delta}^* = N_\phi^2(\underline{\gamma}) = \frac{d}{d\underline{\xi}}[N_\phi^1(\underline{\gamma})]\underline{\phi} = -\,\frac{1}{4}\underline{\gamma}(\underline{\omega}^T\underline{\omega}) + \Gamma(\underline{\gamma})(I_o - I_a)^{-1}(\underline{h}(\underline{\gamma}) x \underline{\omega})$$

$$\Delta = \frac{d}{d\underline{\xi}}[N_\phi^1(\underline{\gamma})]\,\Psi = -\,\Gamma(\underline{\gamma})(I_o - I_a)^{-1}$$

and $\underline{\rho}$ is unchanged. Then the wheel torque, $\underline{\mu}$, is given by

$$\underline{\mu} = (I_o - I_a)\Gamma^{-1}(\underline{\gamma})\,[-\,\frac{1}{4}\,\underline{\gamma}(\underline{\omega}^T\underline{\omega}) + \Gamma(\underline{\gamma})(I_o - I_a)^{-1}(\underline{h}(\underline{\gamma}) x \underline{\omega}) + a_o\underline{\gamma} + a_1\Gamma(\underline{\gamma})\underline{\omega}]$$
$$-\,(I_o - I_a)\Gamma^{-1}(\underline{\gamma})\underline{u}_2$$

which simplifies (somewhat) to

$$\underline{\mu} = -\,\frac{1}{2\gamma_o}(\underline{\omega}^T\underline{\omega} - 4a_o)(I_o - I_a)\underline{\gamma} + \underline{h}(\underline{\gamma}) x \underline{\omega} + a_1(I_o - I_a)\underline{\omega} - (I_o - I_a)\Gamma^{-1}(\underline{\gamma})\underline{u}_2.$$

Definitions of $X_1(\underline{\gamma}, \underline{\omega})$ and $X_2(\underline{\gamma}, \underline{\omega})$ are unchanged from the external torque problem, and

$$\underline{u}_2 = U(\underline{\gamma}, \underline{\omega}, \underline{\mu}) = -\,\frac{1}{4}\underline{\gamma}(\underline{\omega}^T\underline{\omega}) + \Gamma(\underline{\gamma})(I_o - I_a)^{-1}(\underline{h}(\underline{\gamma}) x \underline{\omega})$$

$$= \frac{1}{2\gamma_o}(\underline{\omega}^T\underline{\omega}-4a_o)I_o\underline{\gamma} - I_o\underline{\omega}x\underline{\omega} - a_1I_o\underline{\omega} + I_o\Gamma^{-1}(\underline{\gamma})\underline{u}_1.$$

Notice specifically that $a_o=a_1=0$ gives the same result as found with the Dwyer approach. Again, the transformation is defined as

$$\underline{x}_1 = X_1(\underline{\gamma},\underline{\omega}) = \underline{\gamma}$$

$$\underline{x}_2 = X_2(\underline{\gamma},\underline{\omega}) = \dot{\underline{\gamma}} = \Gamma(\underline{\gamma})\underline{\omega}$$

$$\underline{u}_1 = U(\underline{\gamma},\underline{\omega},\underline{\tau}) = \ddot{\underline{\gamma}} + a_1\dot{\underline{\gamma}} + a_o\underline{\gamma}$$

$$= -\frac{1}{4}\underline{\gamma}(\underline{\omega}^T\underline{\omega}) + \Gamma(\underline{\gamma})I_o^{-1}(I_o\underline{\omega}x\underline{\omega}) + a_1\Gamma(\underline{\gamma})\underline{\omega} + a_o\underline{\gamma} + \Gamma(\underline{\gamma})I_o^{-1}\underline{\tau}.$$

The linear equivalent system is now

$$\dot{\underline{x}} = \begin{bmatrix} 0 & I \\ -a_oI & -a_1I \end{bmatrix}\underline{x} + \begin{bmatrix} 0 \\ I \end{bmatrix}\underline{u}_1$$

where I is again the 3x3 identity matrix and will remain so hereafter. As before, the decoupled nature allows system analysis and controller design to be completed for a scalar version, then generalized to the vector case. Evolution of $\underline{\gamma} = \underline{x}_1$ is a vector equivalent to the single-axis result

$$\ddot{\underline{\gamma}} + a_1\dot{\underline{\gamma}} + a_o\underline{\gamma} = \underline{u}_1.$$

When Freund's technique is applied to the model for internal momentum wheels, much of the work completed for external thrusters can be reused. The elements are defined by

$$N^0_\phi(\underline{\gamma}) = \underline{\gamma}, \qquad \frac{d}{d\underline{\xi}}[N^0_\phi(\underline{\gamma})]\,\Psi = 0$$

$$N^1_\phi(\underline{\gamma}) = \frac{d}{d\underline{\xi}}[N^o_\phi(\underline{\gamma})]\phi = \Gamma(\underline{\gamma})\underline{\omega}, \quad \frac{d}{d\underline{\xi}}[N^1_\phi(\underline{\gamma})]\,\Psi = \Gamma(\underline{\gamma})I_o^{-1} \neq 0, \quad ||\underline{\gamma}||<1$$

and the differential orders d_1, d_2, d_3 are all two. Construction of the elements $\underline{\delta}^*$, Δ, and $\underline{\rho}$ yields

$$\underline{\delta}^* = N^2_\phi(\underline{\gamma}) = \frac{d}{d\underline{\xi}}[N^1_\phi(\underline{\gamma})]\phi = -\frac{1}{4}\underline{\gamma}(\underline{\omega}^T\underline{\omega}) + \Gamma(\underline{\gamma})I_o^{-1}(I_o\underline{\omega}x\underline{\omega})$$

$$\Delta = \frac{d}{d\underline{\xi}}[N^1_\phi(\underline{\gamma})]\,\Psi = \Gamma(\underline{\gamma})I_o^{-1}$$

$$\underline{\rho} = \sum_{k=o}^{1} a_k N^k_\phi(\underline{\gamma}) = a_o\underline{\gamma} + a_1\Gamma(\underline{\gamma})\underline{\omega}.$$

Here the assumption has been made that the choice of free parameters is the same for each of the decoupled channels. If that were not the case, a_o and a_1 would be replaced by diagonal matrices A_o and A_1. This assumption will be used throughout.

Nonlinear input commands are found from

$$\underline{\tau} = -\Delta^{-1}[\underline{\delta}^* + \underline{\rho}] + \Delta^{-1}\underline{u}_1$$

$$= -I_o\Gamma^{-1}(\underline{\gamma})\,[-\tfrac{1}{4}\underline{\gamma}(\underline{\omega}^T\underline{\omega}) + \Gamma(\underline{\gamma})I_o^{-1}(I_o\underline{\omega}x\underline{\omega}) + a_o\underline{\gamma} + a_1\Gamma(\underline{\gamma})\underline{\omega}] + I_o\Gamma^{-1}(\underline{\gamma})\underline{u}_1.$$

Combining terms and using the simplification $\Gamma^{-1}(\underline{\gamma})\underline{\gamma} = \frac{2}{\gamma_o}\underline{\gamma}$

$$\underline{\tau} = \frac{1}{2\gamma_o}I_o\underline{\gamma}(\underline{\omega}^T\underline{\omega}) - I_o\underline{\omega}x\underline{\omega} - \frac{2}{\gamma_o}I_o\underline{\gamma} - a_1 I_o\underline{\omega} + I_o\Gamma^{-1}(\underline{\gamma})\underline{u}_1$$

$$\ddot{\gamma} + a_1\dot{\gamma} + a_o\gamma = u_1.$$

The equivalent linear system is found by setting

$$x_1 = X_1(\gamma,\omega) = \gamma$$

$$x_2 = X_2(\gamma,\omega) = \dot{\gamma} = \frac{1}{2}\sqrt{1-\gamma^2}\,\omega$$

$$u_1 = U(\gamma,\omega,\tau) = \ddot{\gamma} + a_1\dot{\gamma} + a_o\gamma$$

$$= -\frac{1}{4}\gamma\omega^2 + \frac{1}{2}a_1\sqrt{1-\gamma^2}\,\omega + a_o\gamma + \frac{1}{2I_o}\sqrt{1-\gamma^2}\,\tau$$

so that

$$\dot{\underline{x}} = \begin{bmatrix} 0 & 1 \\ -a_o & -a_1 \end{bmatrix} \underline{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_1.$$

Now the free parameters a_o and a_1 can be chosen to provide desired second-order characteristics to the linear system, and thus to the evolution of $\gamma(t)$. A special case is $a_o=a_1=0$. Under this condition the transformation and linear equivalent system reduce to precisely those found using the Dwyer technique.

Setting the vector output $\underline{\delta} = \underline{\gamma}$ in the multi-axis problem also has the desired effect. For the model with external thrusters (11a) and (11b), set $\underline{\xi} = (\underline{\xi}_1^T, \underline{\xi}_2^T)^T = (\underline{\gamma}^T, \underline{\omega}^T)^T$ as before. Then

$$\underline{\phi}(\underline{\xi}) = \begin{bmatrix} \Gamma(\underline{\gamma})\underline{\omega} \\ I_o^{-1}(I_o\underline{\omega}x\underline{\omega}) \end{bmatrix}, \quad \Psi = \begin{bmatrix} 0 \\ I_o^{-1} \end{bmatrix}.$$

Summarizing in vector form the determination of differential orders,

$$\dot{\omega} = \frac{1}{I_o} \tau$$

the components can be identified

$$\underline{\phi}(\underline{\xi}) = \begin{bmatrix} \frac{1}{2} \sqrt{1-\gamma^2}\ \omega \\ 0 \end{bmatrix}, \quad \Psi(\underline{\xi}) = \begin{bmatrix} 0 \\ \frac{1}{I_o} \end{bmatrix}$$

where $\underline{\xi} = (\gamma,\omega)^T$. Consistent with the Dwyer approach, the output is chosen $\delta = \gamma$. First the differential order of γ must be found:

$$N^o_\phi(\gamma) = \gamma, \quad \frac{d}{d\underline{\xi}} [N^o_\phi(\gamma)]\ \Psi = 0$$

$$N^1_\phi(\underline{\gamma}) = \frac{1}{2} \sqrt{1-\gamma^2}\omega, \quad \frac{d}{d\underline{\xi}} [N^1_\phi(\gamma)]\ \Psi = \frac{1}{2I_o} \sqrt{1-\gamma^2} \neq 0 \quad -1<\underline{\gamma}<1.$$

Therefore $d_1 = 2$. Construction of the rest of the elements follows:

$$\delta^* = N^2_\phi(\gamma) = -\frac{1}{4}\gamma\omega^2$$

$$\Delta = \frac{d}{d\underline{\xi}} [N^1_\phi(\gamma)]\ \Psi = \frac{1}{2I_o} \sqrt{1-\gamma^2}, \quad \Delta^{-1} = \frac{2I_o}{\sqrt{1-\gamma^2}}$$

$$\rho = \sum_{k=o}^{1} a_k N^k_\phi(\gamma) = a_o\gamma + a_1(\frac{1}{2} \sqrt{1-\gamma^2}\omega)$$

and

$$\tau = -\Delta^{-1} [\delta^* + \rho] + \Delta^{-1} u_1$$

$$= -\frac{2I_o}{\sqrt{1-\gamma^2}} [-\frac{1}{4}\gamma\omega^1 + a_o\gamma + a_1(\frac{1}{2} \sqrt{1-\gamma^2}\omega] + \frac{2I_o}{\sqrt{1-\gamma^2}} u_1.$$

Since $\delta = \gamma$ is the output, the algorithm says γ will evolve according to

$\underline{\rho}(\underline{\xi})$ where $\rho_i = \sum_{k=o}^{d_i-1} a_{i,k} N_{\phi}^{k} (\delta_i)$ is the i^{th} element.

Then driving the nonlinear system with $\underline{\sigma}$ given by

$$\underline{\sigma} = - \Delta^{-1} [\underline{\delta}^{*} + \underline{\rho}] + \Delta^{-1} \underline{u}$$

results in decoupled outputs δ_i that evolve according to

$$\delta_i^{(d_i)} + \sum_{k=o}^{d_i-1} a_{i,k} \delta_i = u_i$$

where the $a_{i,k}$ are free parameters and $\underline{u}$ is the input to the linear system

$$\dot{\underline{\delta}} = A\underline{\delta} + B\underline{u}.$$

Notice that the success of this approach depends upon the existence of Δ^{-1}. Experience (the hard task-master) has shown that the wrong choice of outputs can result in a Δ matrix which is not invertible at any point in the state space. No discussion is given by Freund to guide the correct choice of outputs when that freedom exists. Again from experience a rule of thumb seems to be to choose a set of elements whose differential orders add to the order of the nonlinear system. This is analogous to condition 1(a) in the Dwyer technique.

For one last time, the single-axis maneuver will serve to illustrate the procedure. With I_o here equal to the scalar moment of inertia, and

$$\dot{\gamma} = \frac{1}{2} \sqrt{1-\gamma^2}\, \omega$$

2.3. Freund Technique

Freund's [12,13] objective was to develop an algorithm to yield nonlinear control laws which would decouple the outputs of nonlinear systems. A side benefit, seemingly, was a description of a linear system involving the outputs and their derivatives. The description of Freund's technique is modified to highlight the result of interest here, the development of an equivalent linear system. In fact, this algorithm as presented formalizes the non-rigorous description of the Dwyer technique with the addition of freely chosen damping terms.

A nonlinear system is described

$$\dot{\underline{\xi}} = \underline{\phi}(\underline{\xi}) + \Psi(\underline{\xi})\underline{\sigma} \qquad \underline{\xi}\varepsilon R^n, \underline{\sigma}\varepsilon R^m.$$

Choose m elements of $\underline{\xi}$, and designate them $\underline{\delta}$, to serve as outputs. Define successive Lie derivative operators recursively by

$$N_{\phi}^{k}(\delta_i) = \frac{d}{d\underline{\xi}}[N_{\phi}^{k-1}(\delta_i)]\underline{\phi}$$

with

$$N_{\phi}^{0}(\delta_i) = \delta_i.$$

Define the differential order of δ_i by

$$d_i = \min\{j : \frac{d}{d\underline{\xi}}[N_{\phi}^{j-1}(\delta_i)]\Psi \neq 0, j=1,2,\ldots,n\}.$$

The algorithm requires the construction of

$$\underline{\delta}^*(\underline{\xi}) \quad \text{where} \quad \delta_i^* = N_{\phi}^{d_i}(\delta_i) \quad \text{is the } i^{th} \text{ element,}$$

$$\Delta(\underline{\xi}) \quad \text{where} \quad \Delta_i = \frac{d}{d\underline{\xi}}[N_{\phi}^{d_i-1}(\delta_i)]\Psi \quad \text{is the } i^{th} \text{ row,}$$

and

$$\underline{\mu} = -\frac{1}{2\gamma_o}(I_o - I_a)^{-1}\underline{\gamma}(\underline{\omega}^T\underline{\omega}) + \underline{h}(\underline{\gamma})x\underline{\omega} - (I_o - I_a)\Gamma^{-1}(\underline{\gamma})\underline{u}_2.$$

Determination of the nonlinear control in each case depends upon the existence of $\Gamma^{-1}(\underline{\gamma})$, which is well defined everywhere except at $\gamma_o = 0$. The condition $\gamma_o \neq 0$ is equivalent to the condition $||\underline{\gamma}|| < 1$ which has been introduced before; therefore, when the operating assumption is satisfied, $U^{-1}(\underline{\gamma},\underline{\omega},\underline{u})$ exists and external or internal torques for the nonlinear system can be determined.

This technique results in identical linear equivalent systems for either internal or external control

$$\dot{\underline{x}} = \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix} \underline{x} + \begin{bmatrix} 0 \\ I \end{bmatrix} \underline{u}$$

where I is the 3x3 identity matrix, $\underline{x} = (\underline{x}_1^T, \underline{x}_2^T)^T$, and $\underline{u}$ is either $\underline{u}_1$ or $\underline{u}_2$. This matrix version of Brunovsky canonical form is equivalent to a set of three decoupled double integrators. The decoupled nature allows controller design to be performed for a scalar model and the result generalized for the vector case. As in the single-axis scenario, the origin of the nonlinear state space corresponds to the origin of the linear state space. Formulation of a control strategy which drives the equivalent linear system to the origin will have the same effect on the nonlinear system.

One drawback tempers (somewhat) enthusiasm about the result of this approach. Poles of the open loop linear equivalent system are at the origin. This can cause a stability problem as will be shown in Chapter Five. The third technique for determining a linear equivalent system remedies this potential problem.

and as before the nonlinear input can be expressed either as $\underline{\tau} = U^{-1}(\underline{\gamma},\underline{\omega},\underline{u}_1)$ or as $\underline{\tau} = W(\underline{x}_1,\underline{x}_2,\underline{u}_1)$. The former is notationally cleaner,

$$\underline{\tau} = \frac{1}{4} I_o \Gamma^{-1}(\underline{\gamma})\underline{\gamma}(\underline{\omega}^T\underline{\omega}) - I_o\underline{\omega}x\underline{\omega} + I_o\Gamma^{-1}(\underline{\gamma})\underline{u}_1$$

where $\Gamma^{-1}(\underline{\gamma})$ can be considered an operator such that

$$\Gamma^{-1}(\underline{\gamma})\underline{u}_1 = \frac{2}{\gamma_o}(\gamma_o^2\underline{u}_1 + \underline{\gamma}\underline{\gamma}^T\underline{u}_1 - \gamma_o\underline{\gamma}x\underline{u}_1)$$

or as a matrix

$$\Gamma^{-1}(\underline{\gamma}) = \frac{2}{\gamma_o}\begin{bmatrix} \gamma_o^2 + \gamma_1^2 & \gamma_o\gamma_3 + \gamma_1\gamma_2 & -\gamma_o\gamma_2 + \gamma_1\gamma_3 \\ -\gamma_o\gamma_3 + \gamma_1\gamma_2 & \gamma_o^2 + \gamma_2^2 & \gamma_o\gamma_1 + \gamma_2\gamma_3 \\ \gamma_o\gamma_2 + \gamma_1\gamma_3 & -\gamma_o\gamma_1 + \gamma_2\gamma_3 & \gamma_o^2 + \gamma_3^2 \end{bmatrix}.$$

A useful result that further simplifies the expression for $\underline{\tau}$ is $\Gamma^{-1}(\underline{\gamma})\underline{\gamma} = \frac{2}{\gamma_o}\underline{\gamma}$. Therefore,

$$\underline{\tau} = \frac{1}{2\gamma_o} I_o\underline{\gamma}(\underline{\omega}^T\underline{\omega}) - I_o\underline{\omega}x\underline{\omega} + I_o\Gamma^{-1}(\underline{\gamma})\underline{u}_1.$$

For the internal momentum wheel model (13a) and (13b), the state transformation $\underline{x} = X(\underline{\gamma},\underline{\omega})$ is left unchanged. However, the expression for $\ddot{\underline{\gamma}}$ must be modified because of the difference in $\dot{\underline{\omega}}$,

$$\underline{u}_2 = \ddot{\underline{\gamma}} = \frac{d}{d\underline{\gamma}}(\Gamma(\underline{\gamma})\underline{\omega})\dot{\underline{\gamma}} + \frac{d}{d\underline{\omega}}(\Gamma(\underline{\gamma})\underline{\omega})\dot{\underline{\omega}}$$

$$= -\frac{1}{4}\underline{\gamma}(\underline{\omega}^T\underline{\omega}) + \Gamma(\underline{\gamma})(I_o-I_a)^{-1}(\underline{h}(\underline{\gamma})x\underline{\omega}) - \Gamma(\underline{\gamma})(I_o-I_a)^{-1}\underline{\mu}.$$

Nonlinear control $\underline{\mu}$, given as $\underline{\mu} = U^{-1}(\underline{\gamma},\underline{\omega},\underline{u})$, is

optimization problem. This difficulty motivated the search for a linear set of state equations that are equivalent, under some change of variables to the nonlinear model.

If an optimal control is now sought by formulating a performance measure which is quadratic in the nonlinear state and transforming that measure into a function of the linear state, the advantage of obtaining linear state equations is then lost. As an example, consider the problem using external thrusters. A cost function is defined

$$J = \frac{1}{2}\, p_1 \underline{\gamma}^T(t_f)\underline{\gamma}(t_f) + \frac{1}{2} p_2 \underline{\omega}^T(t_f)\underline{\omega}(t_f) + \frac{1}{2}\int_{t_o}^{t_f} (q_1 \underline{\gamma}^T(t)\underline{\gamma}(t) + q_2 \underline{\omega}^T(t)\underline{\omega}(t)$$

$$+ r\underline{\tau}^T(t)\underline{\tau}(t))dt.$$

Using the transformation summarized in Table 2.2, this can be expressed in terms of the linear state variables as

$$J = \frac{1}{2} p_1 \underline{x}_1^T(t_f)\underline{x}_1(t_f) + \frac{1}{2} p_2 \underline{x}_2^T(t_f)\Gamma^{-T}(\underline{x}_1(t_f))\Gamma^{-1}(\underline{x}_1(t_f))\underline{x}_2(t_f)$$

$$+ \frac{1}{2}\int_{t_o}^{t_f} (q_1 \underline{x}_1^T(t)\underline{x}_1(t) + q_2 \underline{x}_2^T(t)\Gamma^{-T}(\underline{x}_1(t))\Gamma^{-1}(\underline{x}_1(t))\underline{x}_2(t)$$

$$+ r[\frac{1}{2\gamma_o}(\underline{\omega}^T\underline{\omega}-4a_o)I_o\underline{\gamma} - I_o\underline{\omega}x\underline{\omega} - a_1 I_o\underline{\omega} + I_o\Gamma^{-1}(\underline{\gamma})\underline{u}_1]^T$$

$$\cdot[\frac{1}{2\gamma_o}(\underline{\omega}^T\underline{\omega}-4a_o)I_o\underline{\gamma} - I_o\underline{\omega}x\underline{\omega} - a_1 I_o\underline{\omega} + I_o\Gamma^{-1}(\underline{\gamma})\underline{u}_1]) \, dt,$$

where $\Gamma^{-T}(\underline{\gamma}) = (\Gamma^{-1}(\underline{\gamma}))^T$. Now in place of a quadratic performance measure and nonlinear state equations, the problem has become one of nonquadratic performance measure and linear state equations. This hardly represents a more attractive (or easily solvable) problem.

In many situations the objective of an optimal control approach is to determine a control strategy that will drive the system from its initial state to a desired final state with bounded inputs. (Final time may be either finite or infinite, i.e., specified or unspecified.) The actual form of the cost function may not be the important factor; rather the desire is to find a solution that is easily computable. When this is the case, the performance measure can be formulated as a quadratic function in the linear state,

$$J = \frac{1}{2}\underline{x}^T(t_f)P_f\underline{x}(t_f) + \frac{1}{2}\int_{t_o}^{t_f} (\underline{x}^T(t)Q\underline{x}(t) + \underline{u}^T(t)R\underline{u}(t))dt, \tag{16}$$

and then minimized subject to the linear state equations. The simplicity of the equivalent linear system developed in the previous chapter yields several situations in which an exact analytic expression for the "optimal" control can be found.

For the spacecraft attitude control problem, there are two important conditions which must be satisfied so that a solution to the linear model optimization problem will transform to an acceptable solution to the nonlinear problem. First, a bounded solution for $\underline{u}$ transforms to a bounded solution for $\underline{\sigma}$ only if $\gamma_o \neq 0$ (i.e., $||\underline{\gamma}||<1$). This can be accounted for in the problem statement by the introduction

of an inequality constraint $x_{11}^2+x_{12}^2+x_{13}^2<1$. However, this nonlinear constraint introduces difficulty in solving the problem just as the nonlinear state equations did. Another approach is to solve the unconstrained problem in general, then determine a set of initial conditions for which the solution results in bounded torques. This approach is discussed further in the Chapter Five.

A second requirement is that the terminal state in the linear model must transform to a unique state in the nonlinear model. There is one state which is preserved through both the developed transformation and its inverse. The only state in the nonlinear model that is associated by the transformation with the origin in the linear model is the origin. If the terminal state for reorientation maneuvers can always be defined as the origin, there will be no ambiguity. Driving $\underline{\gamma}$ to the origin is equivalent to aligning the principal axes of the spacecraft with the inertial axes. Since the inertial frame can be arbitrarily chosen, the target state can always be the origin. Further, $\underline{x}_2=0$ implies $\underline{\omega} = \Gamma^{-1}(\underline{\gamma})\underline{x}_2 = \underline{0}$. Therefore, when detumbling is desired (final orientation arbitrary), a solution which drives $\underline{x}_2$ to $\underline{0}$ will result in the spacecraft coming to rest.

There is yet some flexibility in constructing the performance measure (16). Final time can be either finite or infinite. Choice of finite final time results in a trajectory which will be within a neighborhood of the desired terminal state by time t_f. The size of the target area is specified by choice of P_f . Exact terminal control is obtained when $P_f \rightarrow \infty$ (that is, the diagonal elements of P_f become infinite) [23]. Infinite final time results in a trajectory which asymptotically approaches the desired final state. Although the target

state need never be reached in finite time, there is one reason why the final time might be made infinite. Closed loop control with t_f finite results in time varying feedback gains. When $t_f \rightarrow \infty$, the steady state matrix Ricatti equation is solved, rather than the time varying equation. Resulting feedback gains are then constant, and closed loop control, with the associated benefits of state feedback, can be more easily used.

Accepting the premise that the objective of the optimal control problem formulation is to design a trajectory that can be easily computed, the state penalty matrix Q can be set to zero when final time is finite. This allows solution of the time varying Ricatti equation for a gain matrix P(t) to be replaced by solution of a time varying Lyapunov equation for $P^{-1}(t)$. Only first order terms appear in the Lyapunov formulation and the solution is correspondingly easier. If $Q \neq 0$, a solution to the time varying Ricatti equation may be obtained using a method (somewhat more involved) suggested by Juang, Turner and Chun [21].

When the final time is infinite, a positive semi-definite Q will force the state to asymptotically approach the desired terminal state. Then P_f may be set to zero and the steady state Ricatti equation, which is quadratic in the matrix P, must be solved. No benefit is derived in this case from trying to solve for P^{-1} rather than P for the non-zero Q will make the equation in P^{-1} quadratic as well. With $P_f=0$, the state penalty matrix must not also be set to zero for then the performance measure would become $J = \frac{1}{2}\int_{0}^{\infty} \underline{u}^T\underline{u}dt$. With no penalty on the state trajectory or the terminal state, the optimal solution is $\underline{u}=\underline{0}$ and the system will remain in its initial state.

Some advantages can be realized by including a penalty on $\dot{\underline{u}}$ [24]. This is accomplished by defining an augmented state vector $\hat{\underline{x}} = (\underline{x}^T, \underline{u}^T)^T$, increasing the order of P_f and Q appropriately and replacing $\underline{u}$ by $\underline{u}^{(1)} = \dot{\underline{u}}$ in (16). In this situation, the equation $\dot{\underline{u}} = \underline{u}^{(1)}$ is added to the state equations and initial and final values for $\underline{u}$ may be freely chosen.

3.2. Solutions and Simulations

With the equivalent linear system developed in the previous chapter and the foregoing discussion of performance measures in this chapter, an infinite number of linear-quadratic (LQ) optimization problems can be formed. The damping terms must be picked; choice of specified or unspecified final time must be made and if specified, the value must be decided. For finite time problems P_f must be chosen; for infinite time problems Q must be chosen; and the choice of whether or not to penalize $\dot{\underline{u}}$ must be made. For the purposes of example simulations, a desired type of maneuver must be defined with appropriate initial conditions and terminal states. Control actuators (wheels or thrusters) must be chosen, control strategy (open loop or closed loop) must be picked, and the spacecraft parameters (I_o and, if wheels are used, I_a) must be specified. The most general formulation of the problem statement is given by

$$J = \frac{1}{2}p_1\underline{x}_1^T(t_f)\underline{x}_1(t_f) + \frac{1}{2}p_2\underline{x}_2^T(t_f)\underline{x}_2(t_f) + \frac{1}{2}p_3\underline{u}^T(t_f)\underline{u}(t_f)$$

$$+ \frac{1}{2}\int_{t_o}^{t_f} (q_1\underline{x}_1^T(t)\underline{x}_1(t) + q_2\underline{x}_2^T(t)\underline{x}_2(t)$$

$$+ q_3\underline{u}^T(t)u(t) + q_4\dot{\underline{u}}^T(t)\underline{u}(t))dt \tag{17}$$

$$\text{subject to} \quad \dot{\underline{x}}_1 = \underline{x}_2 \qquad \underline{x}_1(0) = \underline{x}_1^o \qquad \underline{x}_1(t_f) = \underline{x}_1^f$$

$$\dot{\underline{x}}_2 = -a_o\underline{x}_1 - a_1\underline{x}_2 + \underline{u} \qquad \underline{x}_2(0) = \underline{x}_2^o \qquad \underline{x}_2(t_f) = \underline{x}_2^f.$$

From this universe of possibilities, five specific cases are selected to illustrate the use of linearizing transformations in spacecraft attitude control. The maneuver is selected to allow comparison to the results of Vadali and Junkins [5]. A slowly tumbling spacecraft is to be stabilized at a specified final orientation. Initial conditions are $\underline{\gamma}(0) = (0.442, 0.442, 0.442)^T$ and $\underline{\omega}(0) = (0.01, 0.005, 0.001)$ with the terminal state being the origin as discussed. The inertial properties are described by

$$I_o = \begin{bmatrix} 87.212 & -0.2237 & -0.2237 \\ -0.2237 & 86.067 & -0.2237 \\ 0.2237 & -0.2237 & 114.562 \end{bmatrix} \quad \text{and} \quad I_a = \begin{bmatrix} 0.05 & 0.00 & 0.00 \\ 0.00 & 0.05 & 0.00 \\ 0.00 & 0.00 & 0.05 \end{bmatrix}.$$

The off-diagonal terms in the system inertia matrix are due to the presence of a spare skew wheel which is not active.

For the purpose of these illustrations, unconstrained input torques are supplied by three momentum wheels with axes along the principal body axes. Perfect knowledge of the system parameters, dynamics, and state and no external disturbances are assumed. This allows an open loop control strategy to be used in each case. However, in the solutions presented the closed loop control law is always found first and could be

used, when appropriate. When finite final time is specified, $P_f \rightarrow \infty$ so that exact terminal control is achieved. The remainder of the specifications are identified as each case is discussed.

Whether damping coefficients are chosen identically for all three channels so that $\dot{\underline{x}}_2 = -a_o\underline{x}_1 - a_1\underline{x}_2 + \underline{u}$, or chosen differently for each channel so that $\dot{\underline{x}} = -A_o\underline{x}_1 - A_1\underline{x}_2 + \underline{u}$ where A_o and A_1 are diagonal matrices, the equivalent linear system is decoupled. Then the entire equivalent linear system can be treated as the combination of three scalar second-order systems. Accordingly, optimization problems are solved in the scalar case and generalized to the vector system. Identical desired response characteristics are assumed for each channel. When the damped system is employed, a_o and a_1 are chosen to provide a critically damped response and are the same for each scalar system. A summary of the five examples follows with specifications referenced to (17). Detailed solutions to the five problems presented are found in Appendix II.

3.2.1 Case I

Finite final time is selected as t_f=100 sec. The norm of the derivative of the input $\underline{u}$ is not penalized, therefore q_4=0 which implies p_3 must also be zero. Terminal penalties p_1 and p_2 are infinite. No penalty is assumed on the norm of the state throughout the trajectory, so q_1=q_2=0. Therefore, the performance measure is given by

$$J_1 = \frac{1}{2}\,\underline{x}^T(t_f)P_f\underline{x}(t_f) + \frac{1}{2}\int_o^{t_f} (\underline{u}^T(t)Ru(t))dt$$

and P_f is allowed to become infinite. The system equations describe a double integrator, therefore $a_o = a_1 = 0$. Solution of this optimization problem yields a closed loop control law

$$\underline{u}_2(t) = -6(t_f - t)^{-2}\, \underline{x}_1(t) - 4(t_f - t)^{-1} \underline{x}_2(t).$$

Analytic solutions can be found for $\underline{x}_1(t)$, $\underline{x}_2(t)$ and $\underline{u}(t)$

$$\begin{aligned} \underline{x}_1(t) &= \underline{c}_1(t_f - t)^3 + \underline{c}_2(t_f - t)^2 \\ \underline{x}_2(t) &= -\, 3\underline{c}_1(t_f - t)^2 - 2\underline{c}_2(t_f - t) \\ \underline{u}_2(t) &= 6\underline{c}_1(t_f - t) + 2\underline{c}_2 \end{aligned}$$

where $\underline{c}_1 = -2t_f^{-3}\underline{x}_1^o - t_f^{-2}\underline{x}_2^o$ and $\underline{c}_2 = 3t_f^{-2}\underline{x}_1^o + t_f^{-1}\underline{x}_2^o$. These expressions are processed through the inverse transformation for use with momentum wheels, and resulting histories for γ_o, $\underline{\gamma}$, $\underline{\omega}$, and $\underline{\mu}$ are shown in Fig. 3.1(a) through (d).

3.2.2 Case II

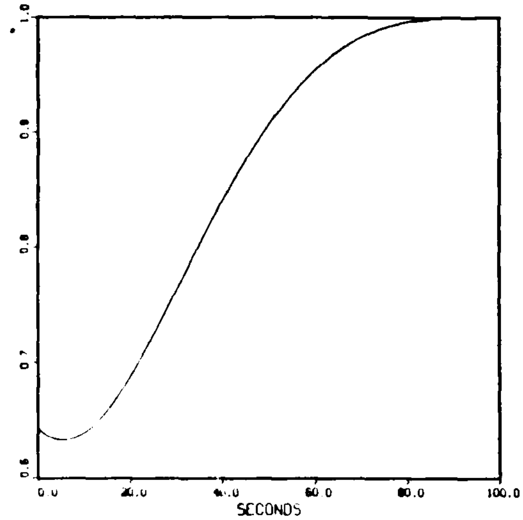
With t_f still 100 secs and a_o and a_1 remaining zero, a penalty is now placed on $||\dot{\underline{u}}||$. Therefore, p_1, p_2, and p_3 are all infinite and q_1, q_2, and q_3 are all zero. The performance measure is now given by

$$J_2 = \frac{1}{2}\, \hat{\underline{x}}^T(t_f) P_f \hat{\underline{x}}(t_f) + \frac{1}{2} \int_o^{t_f} (\dot{\underline{u}}^T(t) R \dot{\underline{u}}(t))dt$$

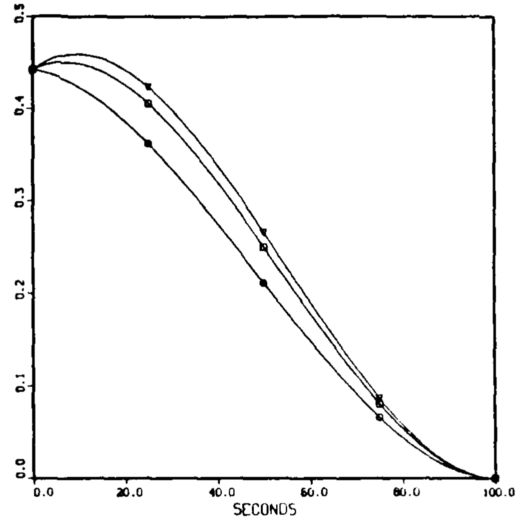
where again $P_f \rightarrow \infty$. This problem results in

$$\dot{\underline{u}}_2(t) = -60(t_f - t)^{-3}\underline{x}_1(t) - 36(t_f - t)^{-2}\underline{x}_2(t) - 9(t_f - t)^{-1}u(t)$$

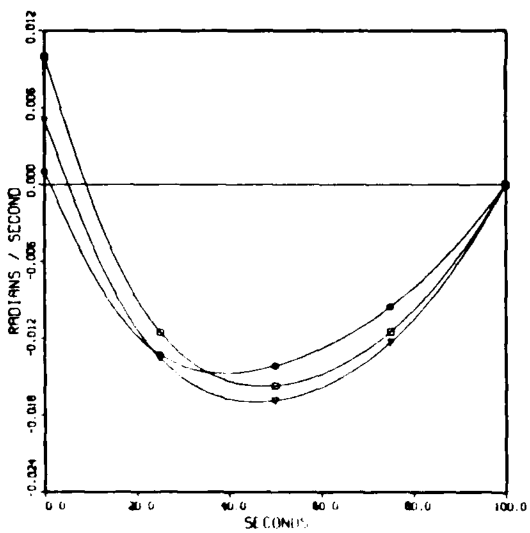
which must then be integrated before being fed through the inverse transformation and applied to the nonlinear system. Expressions for the



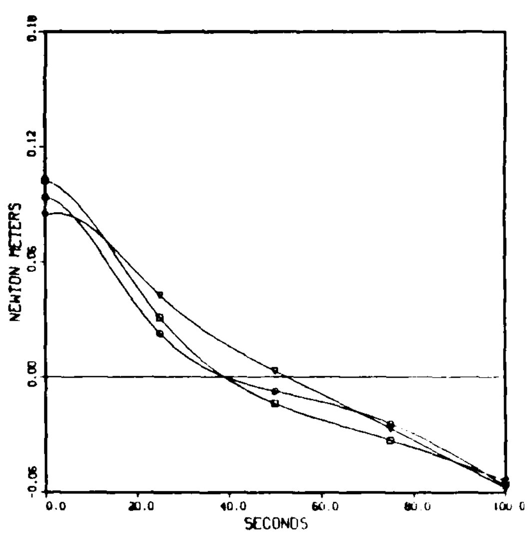
(a) **Attitude Parameter (γ_o)**



(b) **Reduced Euler Parameters ($\underline{\gamma}$)**



(c) **Angular Velocity ($\underline{\omega}$)**



(d) **Applied Torque ($\underline{\mu}$)**

Figure 3.1. Stabilization and reorientation minimizing performance measure J_1, using momentum transfer wheels.
□ = 1st component, ▽ = 2nd component, ○ = 3rd component.

the evolution of $\underline{x}_1$, $\underline{x}_2$ and $\underline{u}$ are

$$\underline{x}_1(t) = \underline{c}_1(t_f-t)^5 + \underline{c}_2(t_f-t)^4 + \underline{c}_3(t_f-t)^3$$
$$\underline{x}_2(t) = -5\underline{c}_1(t_f-t)^4 - 4\underline{c}_2(t_f-t)^3 - 3\underline{c}_3(t_f-t)^2$$
$$\underline{u}_2(t) = 20\underline{c}_1(t_f-t)^3 + 12\underline{c}_2(t_f-t)^2 + 6\underline{c}_3(t_f-t)$$

where $\underline{c}_1 = 6t_f^{-5}\underline{x}_1^o + 3t_f^{-4}\underline{x}_2^o + 0.5t_f^{-3}\underline{u}^o$, $\underline{c}_2 = -15t_f^{-4}\underline{x}_1^o - 7t_f^{-3}\underline{x}_2^o - t_f^{-2}\underline{u}^o$, and $\underline{c}_3 = 10t_f^{-3}\underline{x}_1^o + 4t_f^{-2}\underline{x}_2^o + 0.5t_f^{-1}\underline{u}^o$. Simulations for the nonlinear variables are in Fig. 3.2(a) through (d).

3.2.3 Case III

The third example is an infinite time case. As discussed earlier, p_1, p_2, and p_3 are set to zero. Penalizing the norm of $\underline{u}$ implies $q_3=1$ and $q_4=0$. State penalties q_1 and q_2 are nonzero to drive $(\underline{x}_1,\underline{x}_2)$ to the origin. They are picked by trial and error to be $q_1 = 9x10^{-6}$ and $q_2 = 6x10^{-3}$ to make the $\underline{\gamma}$ trajectory approximate that found in case I and case II. Performance measure J_3 is described by

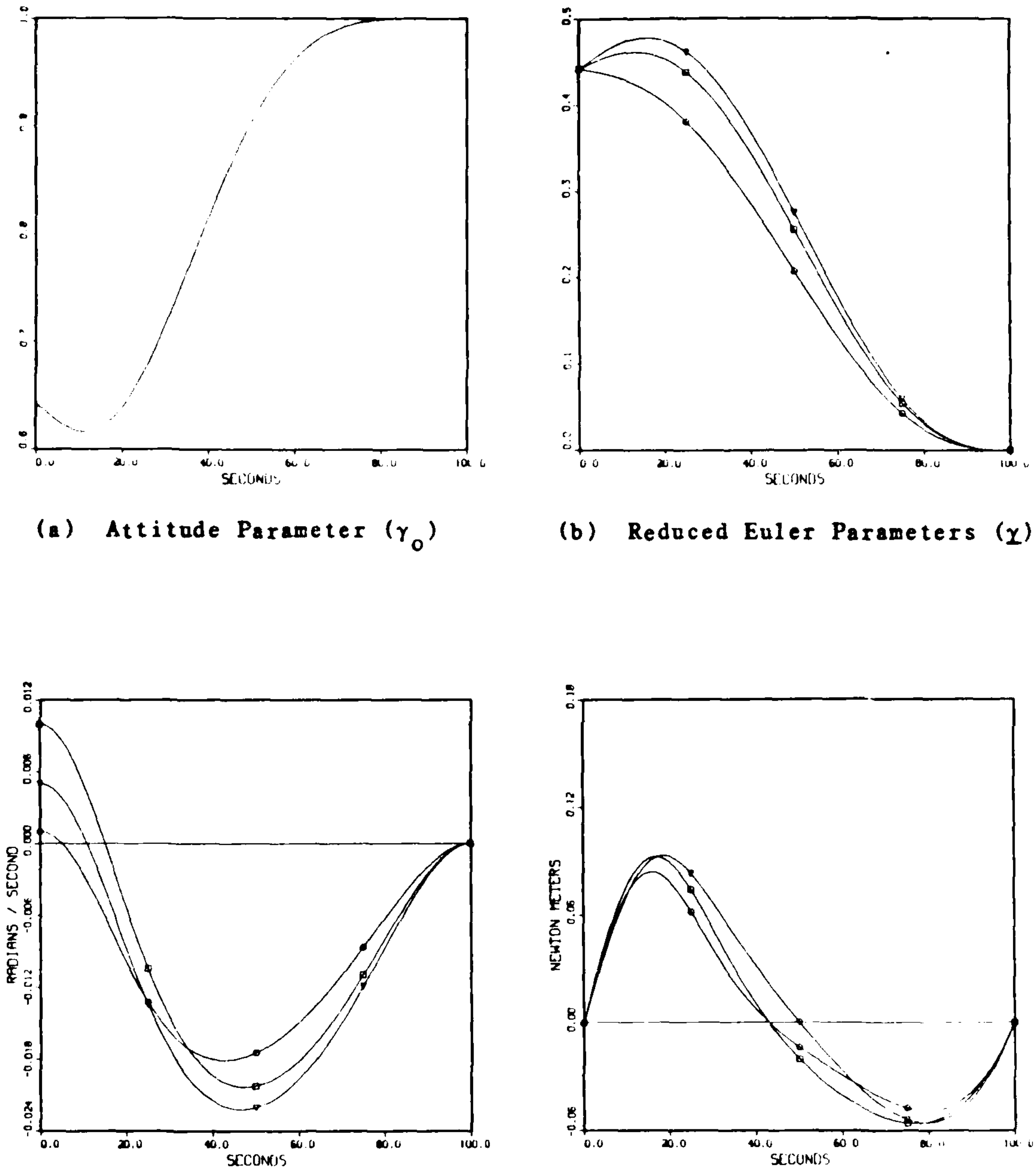
$$J_3 = \frac{1}{2}\int_o^\infty (\underline{x}^T(t)Q\underline{x}(t) + \underline{u}^T(t)\underline{u}(t))dt.$$

Damping terms a_o and a_1 are left zero. The feedback control is given by

$$\underline{u}_2 = -\sqrt{q_1}\,\underline{x}_1 - \sqrt{2\sqrt{q_1} + q_2}\,\underline{x}_2.$$

State trajectories and the open loop control law are represented as

$$\underline{x}_1(t) = \underline{c}_1te^{-\alpha t} + \underline{c}_2e^{-\alpha t}$$
$$\underline{x}_2(t) = -\alpha\underline{c}_1te^{-\alpha t} + (\underline{c}_1 - \alpha\underline{c}_2)e^{-\alpha t}$$
$$\underline{u}_2(t) = \alpha^2\underline{c}_1te^{-\alpha t} - (2\alpha\underline{c}_1 - \alpha^2\underline{c}_2)e^{-\alpha t}$$



(a) Attitude Parameter (γ_o) (b) Reduced Euler Parameters ($\underline{\gamma}$)

(c) Angular Velocity ($\underline{\omega}$) (d) Applied Torque ($\underline{\mu}$)

Figure 3.2. Stabilization and reorientation minimizing performance measure J_2, using momentum transfer wheels.
□ = 1st component, ▽ = 2nd component, ○ = 3rd component.

where $\underline{c}_1 = \alpha\underline{x}_1^o + \underline{x}_2^o$, $\underline{c}_2 = \underline{x}_1^o$, and $\alpha = (q_1)^{1/4}$. Figures 3.3(a) through (d) give the resulting curves for the nonlinear system.

3.2.4 Case IV

An infinite time problem with a penalty on $||\dot{\underline{u}}||$ is solved in case IV. The performance measure in this case is

$$J_4 = \frac{1}{2}\int_0^\infty (\hat{\underline{x}}^T(t)Q\hat{\underline{x}}(t) + \dot{\underline{u}}^T(t)\dot{\underline{u}}(t))dt.$$

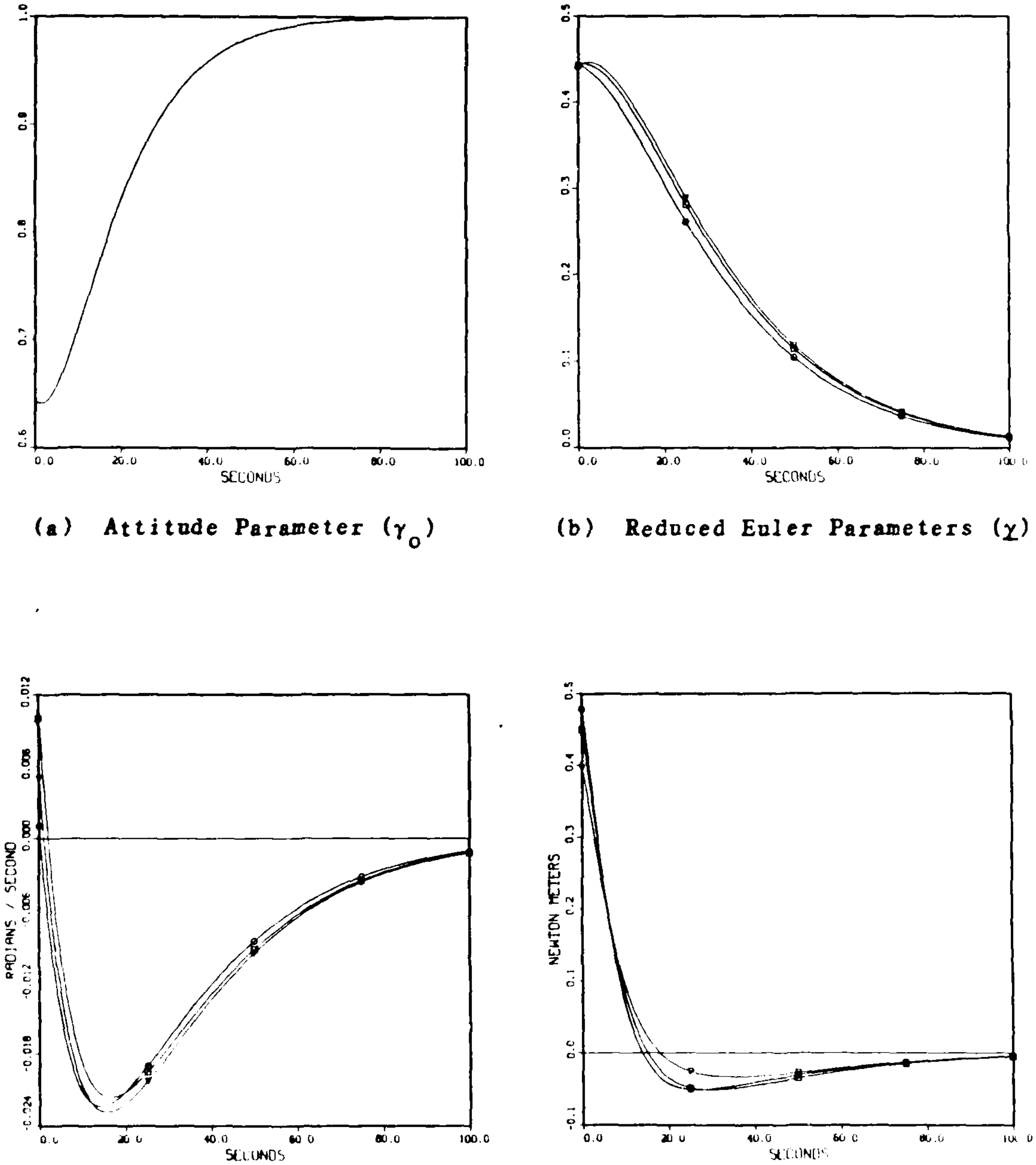
Terminal penalties and damping coefficients are all zero. State penalties q_1 and q_2 and acceleration penalty q_3 are again chosen by trial and error, $q_1 = 1\text{x}10^{-7}$, $q_2 = 3.23\text{x}10^{-4}$ and $q_3 = 1.39\text{x}10^{-2}$, and q_4 is set to 1. The closed loop expression for $\dot{\underline{u}}$ is

$$\dot{\underline{u}}_2(t) = -p_{13}\underline{x}_1(t) - p_{23}\underline{x}_2(t) - p_{33}\underline{x}_3(t)$$

where $p_{13} = \sqrt{q_1}$, p_{23} satisfies $p_{23}^4 - 2q_2p_{23}^2 - 8q_1p_{23} + (q_2^2 - 4q_1q_3) = 0$ and $p_{33} = \sqrt{2p_{23}+q_3}$. (The values for q_1, q_2, and q_3 are picked here such that $p_{13} = \sqrt{q_1} = \alpha^3$, $p_{23} = 3\alpha^2$, and $p_{33} = 3\alpha$.) When $\dot{\underline{u}}$ is integrated and applied to the linear system, the results are

$$\underline{x}_1(t) = \underline{c}_1t^2e^{-\alpha t} + \underline{c}_2te^{-\alpha t} + \underline{c}_3e^{-\alpha t}$$
$$\underline{x}_2(t) = -\alpha\underline{c}_1t^2e^{-\alpha t} + (2\underline{c}_1-\alpha\underline{c}_2)te^{-\alpha t} + (\underline{c}_2-\alpha\underline{c}_3)e^{-\alpha t}$$
$$\underline{u}_2(t) = \alpha^2\underline{c}_1t^2e^{-\alpha t} - (4\alpha\underline{c}_1-\alpha^2\underline{c}_2)te^{-\alpha t} + (2\underline{c}_1-2\alpha\underline{c}_2 + \alpha^2\underline{c}_3)e^{-\alpha t}$$

where $\underline{c}_1 = \frac{1}{2}(\alpha^2\underline{x}_1^o + 2\alpha\underline{x}_2^o + \underline{u}^o)$, $\underline{c}_2 = \alpha\underline{x}_1^o + \underline{x}_2^o$, and $\underline{c}_3 = \underline{x}_1^o$. Associated histories for γ_o, $\underline{\gamma}$, $\underline{\omega}$, and $\underline{\mu}$ are in Figures 3.4(a) through (d).



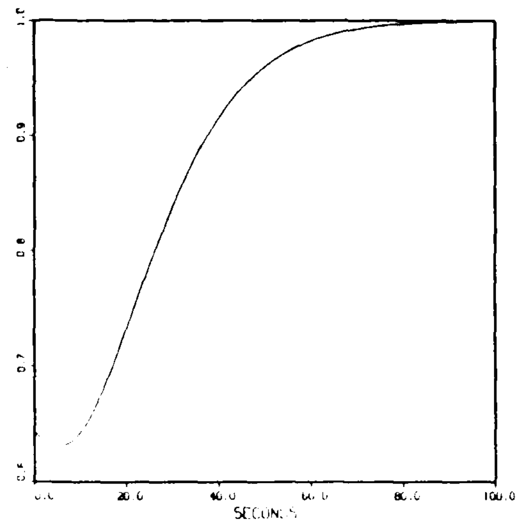
(a) Attitude Parameter (γ_0)

(b) Reduced Euler Parameters ($\underline{\gamma}$)

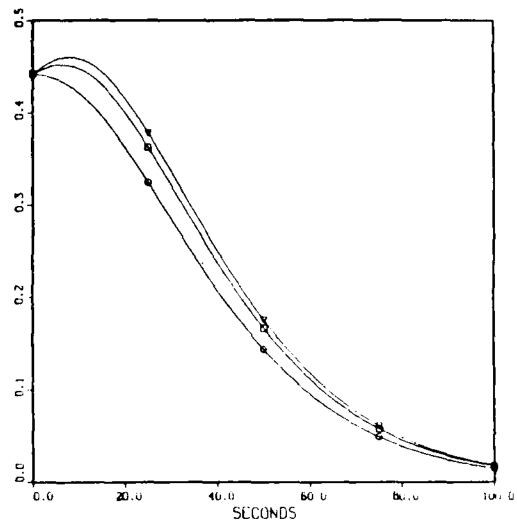
(c) Angular Velocity ($\underline{\omega}$)

(d) Applied Torque ($\underline{\mu}$)

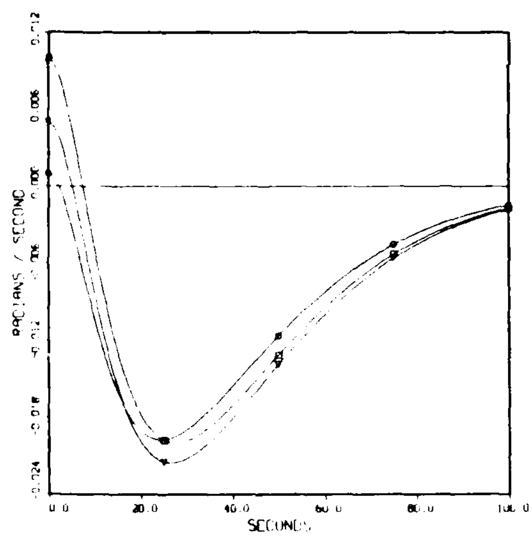
Figure 3.3. Stabilization and reorientation minimizing performance measure J_3, using momentum transfer wheels.
□ = 1st component, ▽ = 2nd component, ○ = 3rd component.



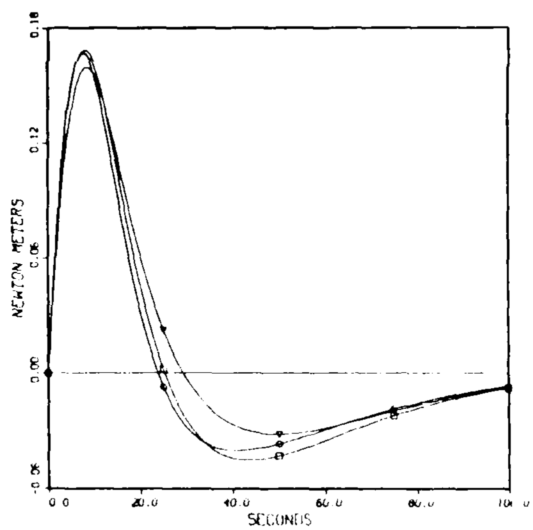
(a) Attitude Parameter (γ_o)



(b) Reduced Euler Parameters ($\underline{\gamma}$)

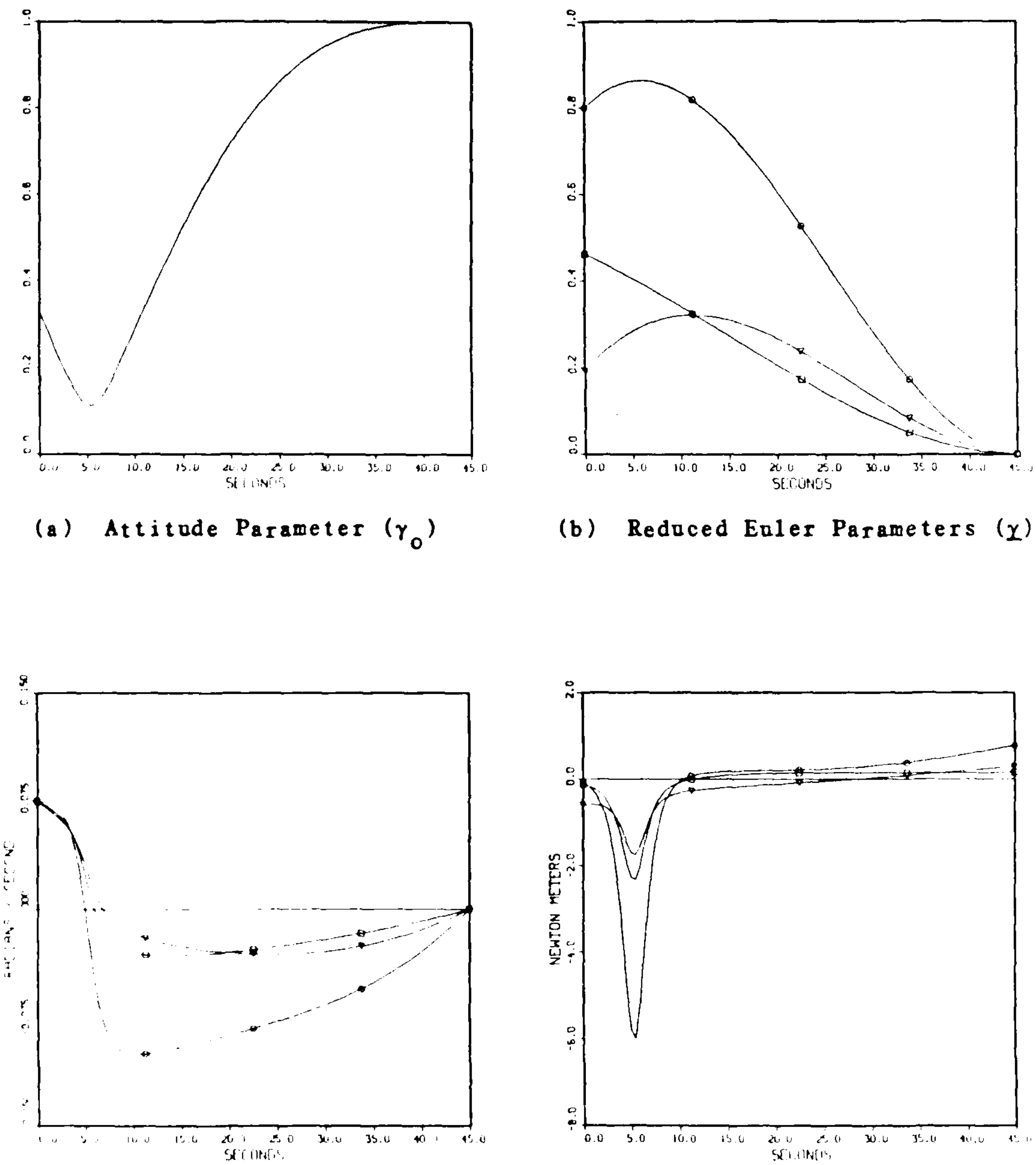


(c) Angular Velocity ($\underline{\omega}$)



(d) Applied Torque ($\underline{\mu}$)

Figure 3.4. Stabilization and reorientation minimizing performance measure J_4, using momentum transfer wheels.
□ = 1st component, ▽ = 2nd component, ○ = 3rd component.



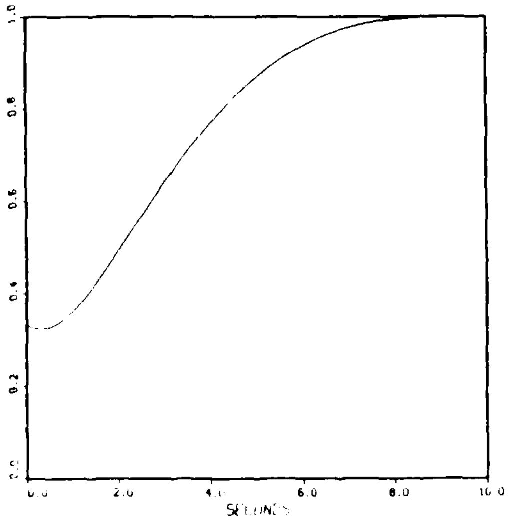
(a) Attitude Parameter (γ_o)

(b) Reduced Euler Parameters ($\underline{\gamma}$)

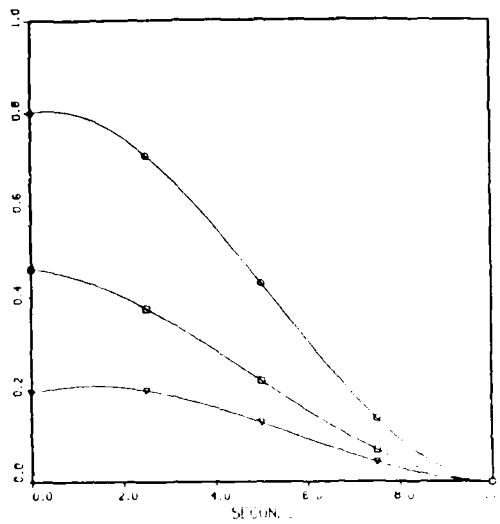
(c) Angular Velocity ($\underline{\omega}$)

(d) Applied Torque ($\underline{\tau}$)

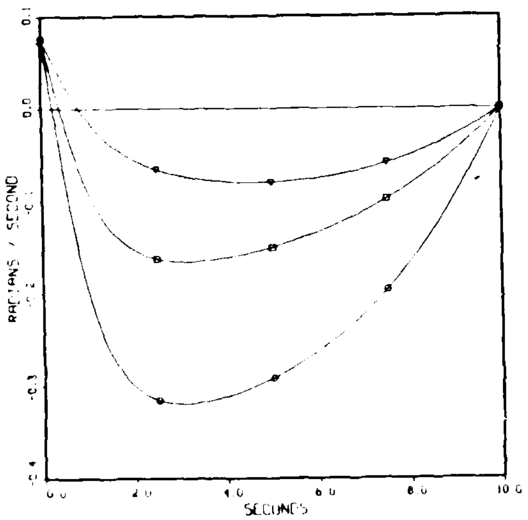
Figure 4.3. Detumbling and orientation minimizing performance measure J_1 with t_f = 45 seconds.
□ = 1st component, ▽ = 2nd component, ○ = 3rd component.



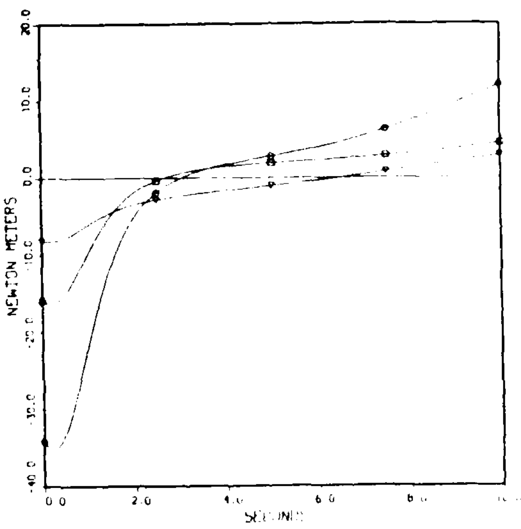
(a) Attitude Parameter (γ_o)



(b) Reduced Euler Parameters ($\underline{\gamma}$)



(c) Angular Velocity ($\underline{\omega}$)



(d) Applied Torque ($\underline{\tau}$)

Figure 4.2. Detumbling and orientation minimizing performance measure J_2 with t_f = 10 seconds.
□ = 1st component, ▽ = 2nd component, ○ = 3rd component.

$$\frac{27}{4\,||\underline{x}_2^o||}(1 - ||\underline{x}_1^o||) = 10.83 \text{ secs}$$

which is less than the 50 second specified final time. But choosing t_1 = 10 secs, the detumbling and orientation maneuver can be accomplished (see Figure 4.2). At $t=t_1$, the input is returned to zero and the spacecraft maintains the desired attitude.

Using (20) to determine an upper limit for maneuver time results in an unnecessarily restrictive bound on t_1. The specific detumbling problem defined earlier can in fact be accomplished if t_1 is chosen to be 45 secs. (see Figure 4.3). Finding this longer acceptable maneuver time allows torque magnitudes to be significantly lowered.

An alternative to attempting to detumble and orient the spacecraft in the same maneuver is to separate the problem into the two distinct tasks; first detumble the spacecraft, then reorient to the desired attitude in a rest-to-rest task. One strategy for detumbling is to set $p_1 = p_3 = q_1 = q_2 = q_4 = 0$ and allow $p_2 \rightarrow \infty$ in the performance index (16), which yields a step function for the acceleration in the linear model,

$$\underline{u}(t) = -\frac{1}{t_1}\underline{x}_2(0).$$

Integrating this expression twice gives

$$\underline{x}_2(t) = (1 - \frac{t}{t_1})\,\underline{x}_2(0)$$

$$\underline{x}_1(t) = \underline{x}_1(0) + (t - \frac{t^2}{2t_1})\,\underline{x}_2(0).$$

Applying the constraint,

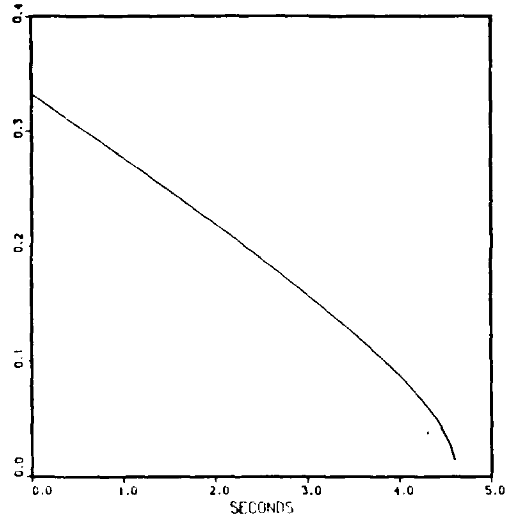
$$||\underline{x}_2^o|| < \frac{27}{4} t_f^{-1} (1 - ||\underline{x}_1^o||). \tag{19}$$

Given the system's initial state, $\underline{x}_1^o$ and $\underline{x}_2^o$, and the desired reorientation time t_f, substitution in (19) to check if the inequality is satisfied is easily accomplished. However, failure to satisfy inequality (19) does not mean the detumbling and orientation cannot be accomplished in the time allotted. In fact, the implication is that too long a time has been specified. This can be seen by rearrangement of the inequality

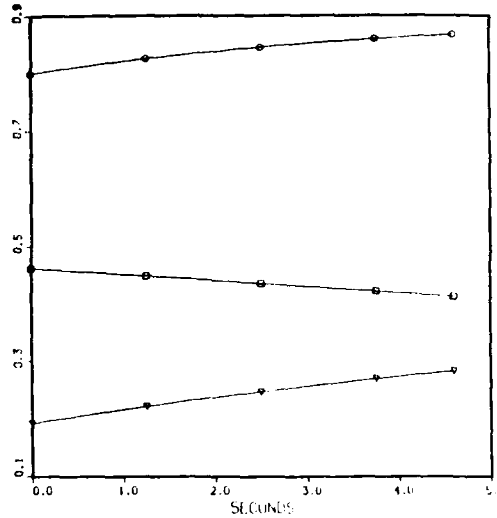
$$t_f < \frac{27}{4\,||\underline{x}_2^o||} (1 - ||\underline{x}_1^o||). \tag{20}$$

Now the bound is on t_f. If t_f as specified satisfies this relationship, then the maneuver can be performed using the control associated with performance measure J_1. If t_f does not satisfy (20), a new maneuver time $t_1 < t_f$ is defined for which the inequality will hold. Final time t_f is replaced in the optimization solutions by t_1 and the desired maneuver is performed. From $t=t_1$ to $t=t_f$, zero input is applied, the spacecraft attitude does not change, and the required orientation is achieved in (less than) the specified time.

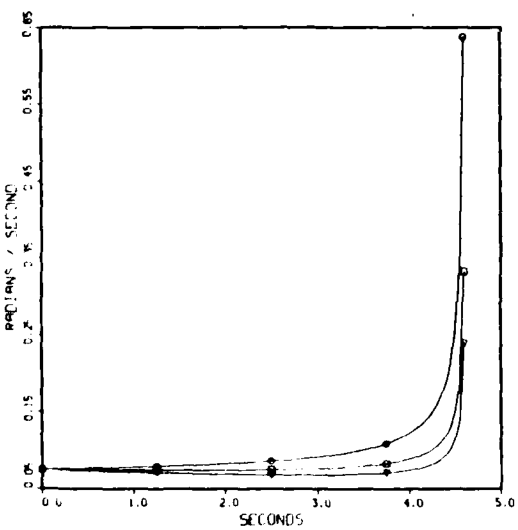
This approach is demonstrated in solving the problem introduced earlier in this section. Inequality (20) (or (19)) is not satisfied with the specifications given. Evaluating the right-hand side of (20) gives



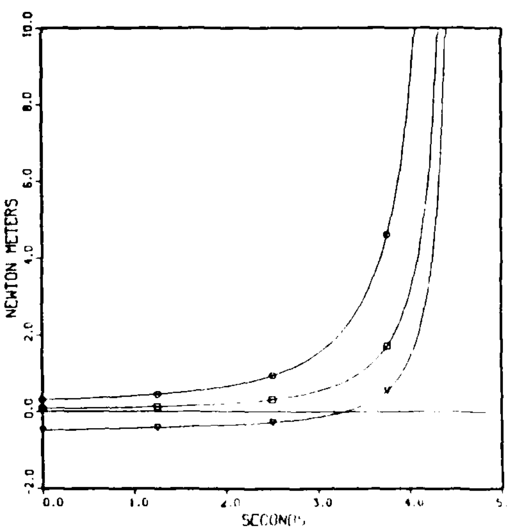
(a) Attitude Parameter (γ_0)



(b) Reduced Euler Parameters ($\underline{\gamma}$)



(c) Angular Velocity ($\underline{\omega}$)



(d) Applied Torque ($\underline{\tau}$)

Figure 4.1. Detumbling and orientation minimizing performance measure J_1 with t_f = 50 seconds.
□ = 1st component, ∇ = 2nd component, ○ = 3rd component.

γ_o becomes 0, $\underline{\gamma}$ becomes unbounded, and the simulation program halts.

Investigating equation (18) further, for $\underline{x}_2^o \neq 0$, sufficient conditions on $\underline{x}_1^o$ and $\underline{x}_2^o$ can be determined under which the singularity will be avoided. From the properties of a norm,

$$||\underline{x}_1(t)|| \leq |-2t_f^{-3}(t_f-t)^3 + 3t_f^{-2}(t_f-t)^2|\cdot||\underline{x}_1^o||$$

$$+ |\ -t_f^{-2}(t_f-t)^3 + t_f^{-1}(t_f-t)^2|\cdot||\underline{x}_2^o||.$$

As shown earlier, the polynomial multiplying $\underline{x}_1^o$ in (18) decreases monotonically from 1 to 0 as t runs from 0 to t_f. A few steps of calculus determine that the polynomial multiplying $\underline{x}_2^o$ is non-negative over the time interval, and has a maximum of $4t_f/27$ at $t=t_f/3$. These results give

$$\max \{|\ -2t_f^{-3}(t_f-t)^3 + 3t_f^{-2}(t_f-t)^2|\ \cdot\ ||\underline{x}_1^o||\} = ||\underline{x}_1^o||\ ,\ 0 \leq t \leq t_f$$

$$\max \{|\ -t_f^{-2}(t_f-t)^3 + t_f^{-1}(t_f-t)^2|\ \cdot\ ||\underline{x}_2^o||\} = \frac{4}{27}t_f\ ||\underline{x}_2^o||\ ,\ 0 \leq t \leq t_f$$

and

$$\max\ ||\underline{x}_1(t)|| \leq ||\underline{x}_1^o|| + \frac{4}{27}\ t_f\ ||\underline{x}_2^o||\ ,\ \ 0 \leq t \leq t_f.$$

The last expression leads to a bound on $||\underline{x}_2^o||$ in terms of $||x_1^o||$ which will guarantee $||\underline{x}_1(t)|| < 1$ for the duration of the maneuver,

For J_2: $p(t) = 6t_f^{-5}(t_f-t)^5 - 15t_f^{-4}(t_f-t)^4 + 10t_f^{-3}(t_f-t)^3$

$$p(0) = 1 \ , \quad p(t_f) = 0$$

$$\frac{d(p(t))}{dt} = -30t_f^{-5}(t_f-t)^2t^2 \leq 0 \qquad \text{for} \qquad 0 \leq t \leq t_f$$

For J_3: $p(t) = (\alpha t+1)e^{-\alpha t}$ (no longer a polynomial)

$$p(0) = 1 \ , \quad p(\infty) = 0$$

$$\frac{d(p(t))}{dt} = -\ \alpha^2 te^{-\alpha t} \leq 0 \qquad \text{for} \qquad 0 \leq t < \infty$$

For J_4: $p(t) = (\frac{1}{2}\alpha^2t^2+\alpha t+1)e^{-\alpha t}$

$$p(0) = 1 \ , \quad p(\infty) = 0$$

$$\frac{d(p(t))}{dt} = -\ \frac{1}{2}\alpha^3t^2e^{-\alpha t}$$

This last expression is non-positive for $0 \leq t < \infty$ if $\alpha \geq 0$. As discussed in Section 3.2.4, α is the cube root of a non-negative number and is therefore non-negative.

Choosing an inertial frame to define $\underline{x}_1(t_f) = 0$ may conceivably result in $||\underline{x}_1^o|| = 1$ (equivalently, $\gamma_o = 0$). The singularity is then excited at t=0. Dividing the maneuver into two rotations, and defining the inertial frame independently for the first rotation, allows the desired reorientation to be accomplished. With this accommodation of the singularity problem at the initial time, any rest-to-rest maneuver may be accomplished without γ_o going to 0.

Such is not the case when the spacecraft is initially tumbling. Consider the problem of detumbling and orienting a spacecraft with external thrusters when the initial states are $\underline{\gamma}(0) = (0.46180,\ 0.19167,\ 0.79987)^T$ and $\underline{\omega}(0) = (0.075,\ 0.075\ 0.075)^T$, and the desired final time is 50 secs. Figure 4.1 shows the simulation results for this problem using performance measure J_1. At t=4.65 sec.,

$$\underline{x}_1(t) = \left[-2t_f^{-3}(t_f-t)^3 + 3t_f^{-2}(t_f-t)^2\right] \underline{x}_1^o$$

$$+ \left[-t_f^{-2}(t_f-t)^3 + t_f^{-1}(t_f-t)^2\right] \underline{x}_2^o. \tag{18}$$

Since $\underline{\omega}(0) = \underline{0}$ implies $\underline{x}_2^o = \underline{x}_2(0) = \Gamma(\underline{\gamma}(0))\underline{\omega}(0) = \underline{0}$, the second term on the right-hand side is zero for all $0 \leq t \leq t_f$. With this simplification, $\underline{x}_1(t) = p(t)\underline{x}_1^o$ where the polynomial $p(t) = -2t_f^{-3}(t_f-t)^3 + 3t_f^{-2}(t_f-t)^2$ has the properties $p(0) = 1$ and $p(t_f) = 0$. Consider the time derivative of $p(t)$,

$$\frac{d(p(t))}{dt} = 6t_f^{-3}(t_f-t)^2 - 6t_f^{-2}(t_f-t)$$

$$= 6t_f^{-3}(t_f-t)\ [(t_f-t) - t_f]$$

$$= 6t_f^{-3}(t_f-t)\ [-t].$$

The three terms outside the bracket are all non-negative and the term $[-t]$ is non-positive for $0 \leq t \leq t_f$. Therefore, the derivative of $p(t)$ is non-positive and $p(t)$ is monotone decreasing for the duration of the maneuver. Because of the nature of this polynomial, $||\underline{x}_1(t)|| = |p(t)| \cdot ||\underline{x}_1^o|| \leq |p(o)| \cdot ||\underline{x}_1^o|| = ||\underline{x}_1^o||$ for all $0 \leq t \leq t_f$. If the initial orientation of the spacecraft does not correspond to $\gamma_o = 0$, or equivalently $||\underline{x}_1^o|| = 1$, the singularity will not be encountered in this rest-to-rest maneuver. An argument similar to that just presented can be developed for the remaining performance measures.

This chapter addresses the problems of singularity avoidance and variation of parameters. Control laws under which $||\underline{\gamma}(t)|| < 1$ for all $0 \leq t \leq t_f$ are developed. Compensation for variations from the modeled inertial properties is introduced. Three approaches to the control of spacecraft attitude with torque actuator constraints are discussed in Chapter Five. Simulations demonstrate the capability to deal with these three difficulties.

4.1. Singularity Avoidance

In determining conditions under which the transformation singularity will not be excited, benefits are realized from dividing the spacecraft attitude control problem into two sub-tasks. A rest-to-rest maneuver is characterized by $\underline{\omega}(0) = \underline{0}$ and $\underline{\omega}(t_f) = \underline{0}$. The inertial frame is chosen as discussed in Chapter Three to define the desired position parameter as $\underline{\gamma}(t_f) = \underline{0}$. Detumbling is characterized by $\underline{\omega}(0) \neq \underline{0}$, but $\underline{\omega}(t_f) = \underline{0}$. Initial and final values of the Euler parameters are determined by choice of the inertial frame and the control strategy.

The advantage of this division of tasks is realized in considering the rest-to-rest maneuver. If $||\underline{\gamma}(0)|| < 1$ and $||\underline{\gamma}(t)|| \leq ||\underline{\gamma}(0)||$ for all $0 \leq t \leq t_f$, then γ_o will never be zero and the singularity will be avoided. Since the transformations developed in Chapter Two define $\underline{x}_1 = \underline{\gamma}$, and since closed form expressions are determined for $\underline{x}_1$ in Chapter Three, the condition on $||\underline{\gamma}(t)||$ can be readily tested. (Throughout the following discussion, $\underline{x}_1$ and $\underline{\gamma}$ are used interchangeably.)

When performance measure J_1 (as discussed in Section 3.2.1) is minimized, the parametric attitude evolves according to

CHAPTER FOUR

DEALING WITH DIFFICULTIES I

Solutions of the optimization problems in the previous chapter, and the associated simulations, are developed under a few assumptions which limit practical application to the spacecraft attitude control problem. Two constraints on the Euler parameters are ignored. The identity $\gamma_o^2 + ||\underline{\gamma}||^2 = \gamma_o^2+\gamma_1^2+\gamma_2^2+\gamma_3^2 = 1$ is physically satisfied at every instant of time, but no provision for this is made mathematically. A singularity exists in the input transformation from the linear model to the nonlinear model when $\gamma_o = 0$. This further constrains the norm of $\underline{\gamma}$ to $||\underline{\gamma}|| < 1$. A second assumption inherent in the solution is that the plant is perfectly known. Inertial properties of the spacecraft and wheels are the only plant parameters to appear in the mathematical model. Variation of actual values of those parameters from the values assumed in the model act as a disturbance and perturb the actual state trajectory from the nominal determined in Chapter Three. Finally, stated in the solutions in the previous chapter is the assumption that input torques have no physical constraints, or the constraints are so large that they may be ignored. When inputs commanded by the solution of the unconstrained optimization problems exceed actuator capabilities, a clipped version of the desired torque will be applied, and the spacecraft attitude history will not follow the intended path.

ignored. Since $\underline{x}_1 = \underline{y}$, this constraint implies $||\underline{x}_1(t)|| \leq 1$ for all t. Further, to avoid the singularity in the inverse transformation from $\underline{u}_2$ to $\underline{u}$, the strict inequality $||\underline{x}_1(t)|| < 1$ must hold. Satisfaction of these constraints, and compensating for possible variations in system parameters are discussed in the next chapter.

for $\underline{u}_2$ in problem III. When $q_1=q_2=0$, $\underline{u}_2 = -\sqrt{q_1}\,\underline{x}_1 - \sqrt{2\sqrt{q_1}+q_2}\,\underline{x}_2 = \underline{0}$ as stated. A similar answer is found if q_1, q_2, and q_3 are set to zero in case IV.

A linear differential equation for $\underline{x}_1(t)$ can be found when the system equations and the closed loop control law are known. For case III, this differential equation is

$$\ddot{\underline{x}}_1(t) + \sqrt{2\sqrt{q_1}+q_2}\,\dot{\underline{x}}_1(t) + \sqrt{q_1}\,\underline{x}_1(t) = 0$$

The state penalties can be chosen to give a critically damped response curve for $\underline{x}_1(t)$. Similarly in case V,

$$\ddot{x}_1(t) + \sqrt{a_1^2-2a_o+2\sqrt{a_o^2+q_1}+q_2}\,\dot{\underline{x}}_1(t) + \sqrt{a_o^2+q_1}\,\underline{x}_1(t) = 0.$$

Now a_o, a_1, q_1, and q_2 can be picked (a_o, a_1 nonzero) such that the response is critically damped. A significant difference exists between the two cases although the trajectories look identical. The system described in case V is damped; the system of case III is not. Open loop system poles for case V have negative real parts, whereas open loop system poles for case III are at the origin.

An important point is that the same solutions yielding $\underline{u}_2$ for wheel control could be used to generate the linear acceleration $\underline{u}_1$ for variable thruster control, when available.

The purpose of the five examples presented here is to demonstrate the feasibility of solving the nonlinear spacecraft attitude control problem using the linearizing transformation approach. However, to this point, the constraint on the Euler parameters, $\gamma_o^2+\gamma_1^2+\gamma_2^2+\gamma_3^2 = 1$, has been

3.3 Discussion

The point must be emphasized that closed loop control is not appropriate for cases I and II. As t approaches the final time, the feedback gains increase without bound. Even though the states $\underline{x}_1$ and $\underline{x}_2$ are going to zero and the result is in fact a bounded acceleration $\underline{u}$, the gains cannot be implemented. If feedback control is desired, then some scheme to work around the end point singularity must be found.

The effect of penalizing $||\dot{\underline{u}}(t)||$ in addition to, or instead of, penalizing $||\underline{u}(t)||$ is to allow free choice for initial and final values of $\underline{u}$. This can be considered a first attempt at "acceleration shaping." For finite final time, the expression for $\underline{u}_2$ changes from linear to cubic in time as this penalty is added. A corresponding change is noted in the curves shown for $\underline{u}$ as desired. In case I, the torque is basically a monotone decreasing function, whereas in case III, $\underline{u}$ has the general shape of a distorted third-order polynomial. The input to the nonlinear system can be informally considered a distorted version of the input to the linear system.

There is little difference in the maximum torque required in each channel as the penalty shifts from $||u||$ to $||\dot{u}||$ with finite final time. But for the infinite time problem, a decrease of the order of 60% of the maximum is shown when penalizing $||\dot{\underline{u}}||$. (This appears to be contradictory to Murphy's law. A smoother torque profile is generated, the maximum required torque is considerably smaller, and the state trajectories are still acceptable.)

Earlier the point was made that for infinite time problems, if the terminal state penalities and the state trajectory penalties are both zero, the optimal solution is $\underline{u}=\underline{0}$. This is affirmed in the expression

3.2.5 Case V

System damping is introduced in the final case. Final time is infinite and the penalty on $\dot{\underline{u}}$ is dropped. Therefore p_1, p_2, p_3, and p_4 are zero, and $q_3=1$. The performance measure becomes

$$J_5 = \frac{1}{2}\int_0^\infty (\underline{x}^T(t)Q\underline{x}(t) + \underline{u}^T(t)\underline{u}(t))dt$$

with system equations

$$\dot{\underline{x}}_1 = \underline{x}_2$$

$$\dot{\underline{x}}_2 = -a_o\underline{x}_1 - a_1\underline{x}_2 + \underline{u}_2$$

where a_o and a_1 are nonzero. Solution of the optimization problem for general a_o, a_1, q_1, and q_2 results in

$$\underline{u}_2(t) = (a_o - \sqrt{a_o^2+q_1})\ \underline{x}_1(t) + (a_1 - \sqrt{a_1^2-2a_o+2\sqrt{a_o^2+q_1} + q_2})\ \underline{x}_2(t).$$

Recalling that a_o and a_1 are free parameters introduced in the transformation to an equivalent linear system, they may be chosen along with q_1 and q_2 to give the identical response achieved in case III. Letting q_1' and q_2' represent the state penalties from the earlier case, set $q_1 = a_o^2 = \frac{1}{2} q_1'$ and $q_2 = a_1^2 = \frac{1}{2} (\sqrt{2q_1'} + q_2')$. Analytic expressions for $\underline{x}_1(t)$, $\underline{x}_2(t)$, and $\underline{u}(t)$ and simulations for the variables in the nonlinear model are then the same as in example III and are not repeated here.

$$||\underline{x}_1(t)|| = ||\underline{x}_1(0) + (t - \frac{t^2}{2t_1})\,\underline{x}_2(0)||$$

$$\leq ||\underline{x}_1(0)|| + |t - \frac{t^2}{2t_1}| \cdot ||\underline{x}_2(0)||$$

$$\leq ||\underline{x}_1(0)|| + \frac{1}{2}\,t_1||\underline{x}_2(0)|| < 1, \quad 0 \leq t \leq t_1$$

yields

$$t_1 < \frac{2}{||\underline{x}_2(0)||}\,(1 - ||\underline{x}_1(0)||).$$

At first consideration, this seems to be an even more restrictive limit on t_1 than the previous strategy. However, since two separate tasks are defined, the flexibility in choosing the inertial frame can be used to increase the bound on t_1. For the detumbling portion of the maneuver, choose the inertial frame such that $\underline{x}_1(0) = \underline{\gamma}(0) = \underline{0}$, and $\underline{x}_2(0) = \Gamma(\underline{\gamma}(0))\underline{\omega}(0) = \underline{\omega}(0)$. The bound on t_1 becomes $t_1 < 2/||\underline{\omega}(o)||$, and with the initial values given, $t_1 < 15.40$ sec. If t_1 is chosen to be 15.0 secs, detumbling occurs as t runs from 0 to 15.0, the inertial frame is redefined at t=15.0, and reorientation is performed from t=15.0 to t=50.0.

Under the continuing assumption of unconstrained inputs, the problem of the singularity in the input transformation is successfully dealt with. The singularity is encountered in rest-to-rest maneuvers only as an initial condition. When this occurs, the task can be accomplished by dividing it into two reorientation sub-tasks. Bounds are given for maneuver times for either tumbling-to-orientation or strictly detumbling tasks which will maintain $\gamma_o \neq 0$.

A side effect of avoiding the singularity is satisfaction of the constraint on the Euler parameters. Since $||\underline{\gamma}|| < 1$, and since γ_o is not being directly controlled, the physical constraint $\gamma_o^2+\gamma_1^2+\gamma_2^2+\gamma_3^2 = 1$ determines the manner in which γ_o will evolve.

4.2 Variation of System Parameters

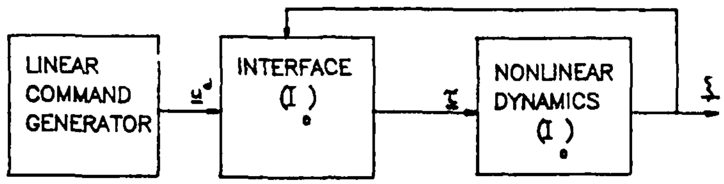
Perfect knowledge of the spacecraft and wheel inertial properties is assumed in the previous chapter. However, as expendable fuel for external thrusters is used, the matrix I_o will change to some extent. Variations in I_o will affect not only the system dynamics, but also the interface between the linear acceleration command generator and the torque actuator. If compensation for these variations is not provided, the actual state trajectory will not end in the desired terminal orientation. This situation is illustrated in Figure 4.4(a) and (b).

Using the double integrator linear model, the command generator sends the desired linear acceleration $\underline{u}^d = (\ddot{\underline{\gamma}})^d$ to the interface. The assumed value of the inertial matrix is used in the interface to transform from $\underline{u}^d$ to $\underline{\tau}^d$. Applying $\underline{\tau}^d$ to the system dynamics when I_o is indeed known results in the desired histories $\underline{\gamma}^d$ and $\underline{\omega}^d$. But when the inertial properties have changed, from I_o to I_o', the desired trajectories are no longer followed. This is shown mathematically by comparing $(\ddot{\underline{\gamma}})^d$ and $\ddot{\underline{\gamma}}$:

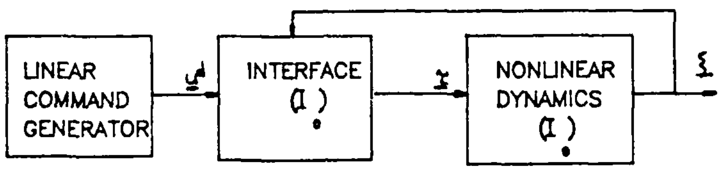
$$\underline{u}^d = (\ddot{\underline{\gamma}})^d = -\frac{1}{4}\left((\underline{\omega}^d)^T\underline{\omega}^d\right)\underline{\gamma}^d + \Gamma(\underline{\gamma}^d)I_o^{-1}(I_o\underline{\omega}^d \times \underline{\omega}^d) + \Gamma(\underline{\gamma}^d)I_o^{-1}\underline{\tau}^d$$

but

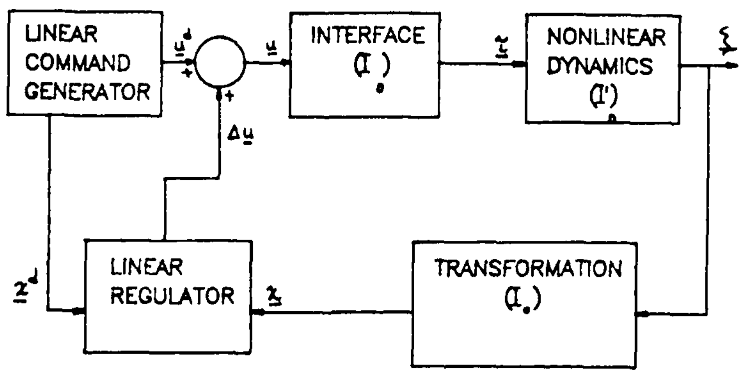
$$\ddot{\underline{\gamma}} = -\frac{1}{4}(\underline{\omega}^T\underline{\omega})\underline{\gamma} + \Gamma(\underline{\gamma})(I_o')^{-1}(I_o'\underline{\omega} \times \underline{\omega}) + \Gamma(\underline{\gamma})(I_o')^{-1}\underline{\tau}^d.$$



(a) Modeled and actual I_o agree; desired trajectory results.



(b) Modeled and actual I_o disagree; perturbed trajectory results.



(c) Correction in the linear model for variation from modeled I_o.

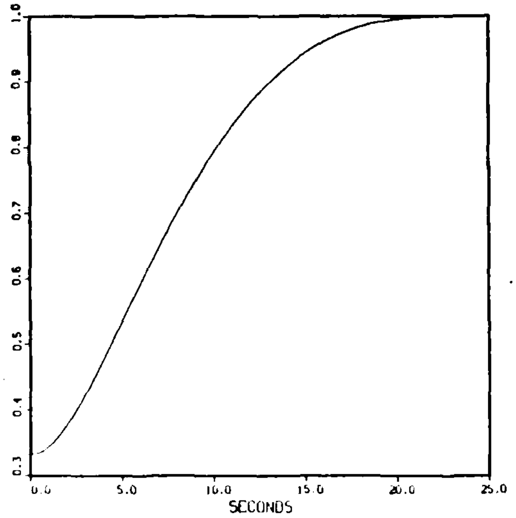
Figure 4.4. Effect of imperfect knowledge of spacecraft inertial properties on state trajectories.

An example demonstrates the effect of the change of inertial parameters on the state trajectories. A rest-to-rest maneuver is defined with initial orientation represented by $\underline{\gamma}(0) = (0.46180, 0.19167, 0.79987)^T$ and desired terminal orientation $\underline{\gamma}(t_f) = \underline{0}$. Final time is specified as 25 seconds. External thrusters are to be used to accomplish the reorientation. Inertial properties are assumed to be the same as in Chapter Three. Performance measure J_1 is used to derive the desired control strategy. The nominal trajectory simulation is shown in Figure 4.5. When a 20% perturbation in moments and products of inertia is experienced in the system dynamics, but not in the interface, the trajectories diverge from the desired paths (Figure 4.6).

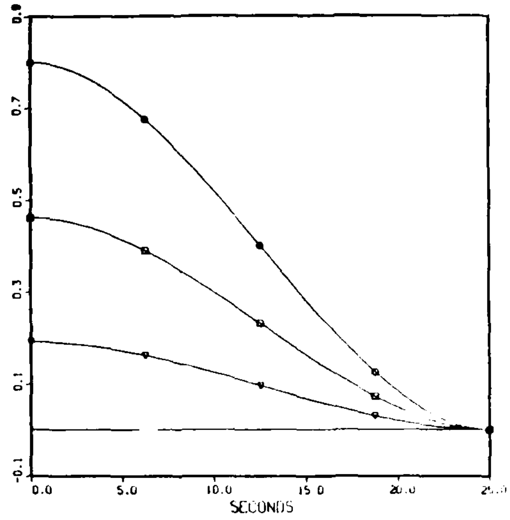
Correction for perturbations in system parameters is accomplished by introducing a $\Delta\underline{u}$ to the desired input $\underline{u}^d$[25]. This correction is generated by applying a linear regulator to the equivalent linear system's desired and actual state histories (see Figure 4.4 (c)). Desired linear system state histories are supplied by the command generator. Actual states of the equivalent linear system are obtained by transforming the measured physical states $\underline{\gamma}$ and $\underline{\omega}$. (Assumed values of I_o are used in this transformation since the actual I_o' is unknown.) The error state equations for regulator design are given by

$$\Delta\dot{\underline{x}} = \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix} \Delta\underline{x} + \begin{bmatrix} 0 \\ I \end{bmatrix} \Delta\underline{u} + \begin{bmatrix} 0 \\ I \end{bmatrix} \underline{w}$$

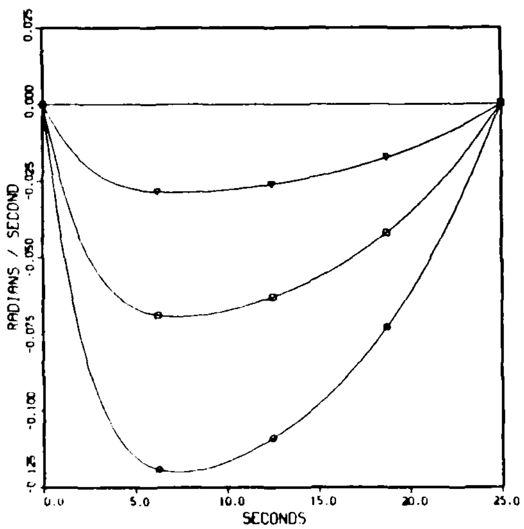
where $\Delta\underline{x} = \underline{x} - \underline{x}^d$, $\Delta\underline{u} = \underline{u} - \underline{u}^d$, and $\underline{w} = \dot{\underline{x}}_2 - \underline{u}$. As before, $\dot{\underline{x}}_2 \neq \underline{u}$ because of the difference between I_o in the transformation and I_o' in the dynamics.



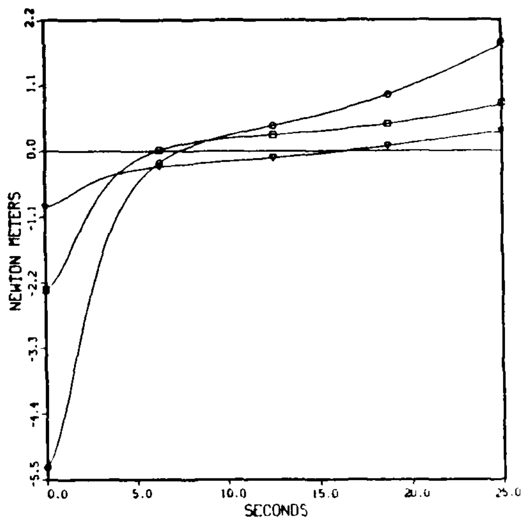
(a) Attitude Parameter (γ_o)



(b) Reduced Euler Parameters ($\underline{\gamma}$)

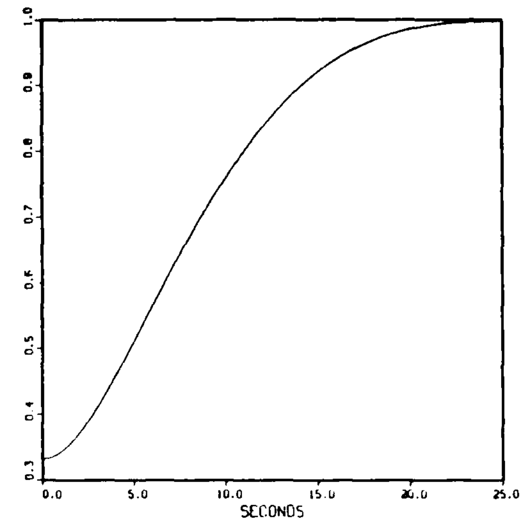


(c) Angular Velocity ($\underline{\omega}$)

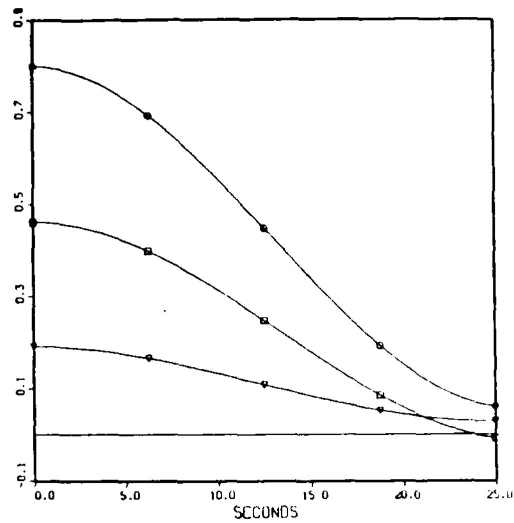


(d) Applied Torque ($\underline{\tau}$)

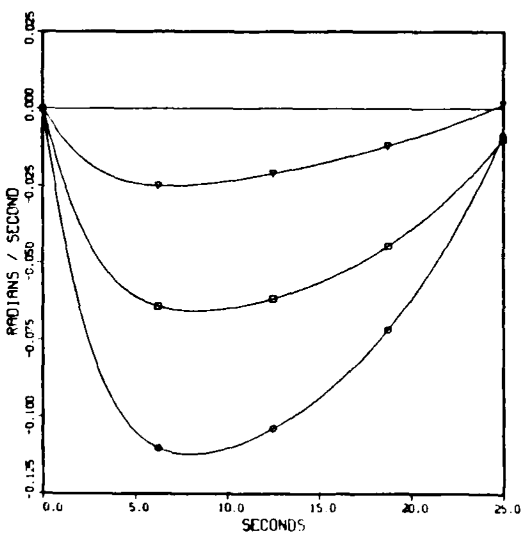
Figure 4.5. Nominal state and input histories for reorientation with perfect knowledge of I_o.
□ = 1st component, ▽ = 2nd component, ○ = 3rd component.



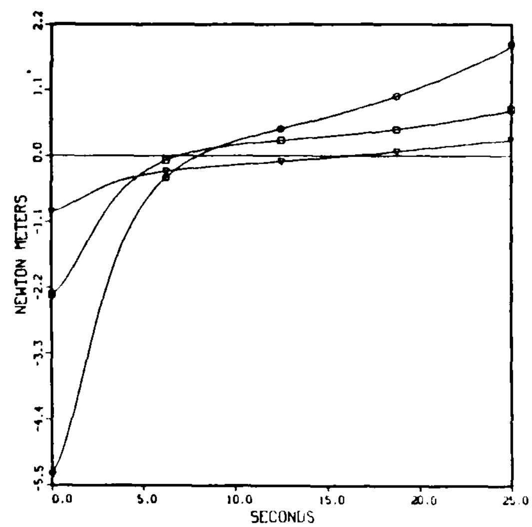
(a) Attitude Parameter (γ_o)



(b) Reduced Euler Parameters ($\underline{\gamma}$)



(c) Angular Velocity ($\underline{\omega}$)



(d) Applied Torque ($\underline{\tau}$)

Figure 4.6. Perturbed state and input histories for reorientation due to imperfect knowledge of I'_o.
□ = 1st component, ▽ = 2nd component, ○ = 3rd component.

A first attempt at finding $\Delta \underline{u}$ assumes $\underline{w}$ to be unwanted noise. Modifying the performance measure J_1 appropriately, $\Delta \underline{u}$ is determined by minimizing

$$J = \frac{1}{2} p_1 (\Delta \underline{x}_1(t_f))^T \Delta \underline{x}_1(t_f) + \frac{1}{2} p_2 (\Delta \underline{x}_2(t_f))^T (\Delta \underline{x}_2(t_f)) + \frac{1}{2} \int_0^{t_f} (\Delta \underline{u}(t))^T (\Delta \underline{u}(t)) dt$$

subject to the above state equations. Solution of this optimization problem parallels that given in Appendix B.1, but to use feedback to account for noise while avoiding infinite gains at $t=t_f$, the terminal state penalties p_1 and p_2 are left finite. Actual values of p_1 and p_2 are adjusted to achieve best results. The correction to the desired linear control is then

$$\Delta \underline{u} = - p_{12} \Delta \underline{x}_1 - p_{22} \Delta \underline{x}_2$$

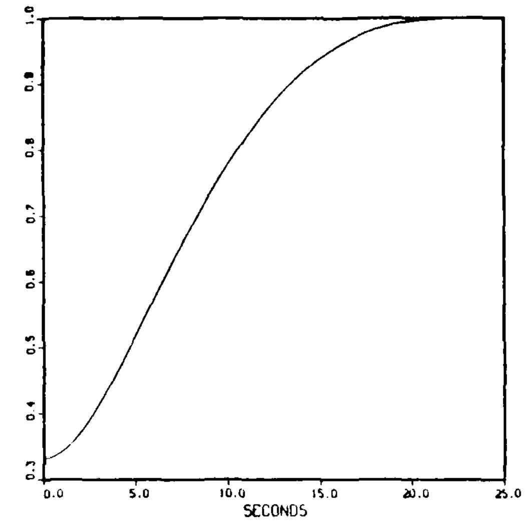
where $p_{12} = (\frac{1}{2} p_1 p_2 (t_f - t)^2 + p_1 (t_f - t))/d(t)$,

$$p_{22} = (\frac{1}{3} p_1 p_2 (t_f - t)^3 + p_1 (t_f - t)^2 + p_2)/d(t)$$

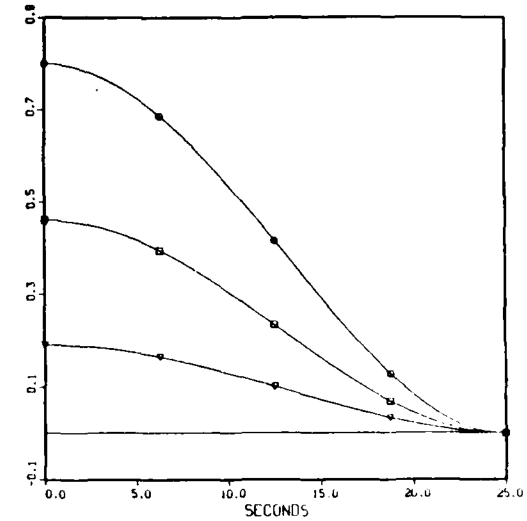
and $$d(t) = \frac{1}{12} p_1 p_2 (t_f - t)^4 + \frac{1}{3} p_1 (t_f - t)^3 + p_2 (t_f - t) + 1.$$

Simulated results for this approach to correction for variation in system parameters are shown in Figure 4.7. Values used for the terminal state penalties are p_1 and p_2 both 10^4. Assuming $\underline{w} = \dot{\underline{x}}_2 - \underline{u}$ to be noise appears not to have significantly deleterious results.

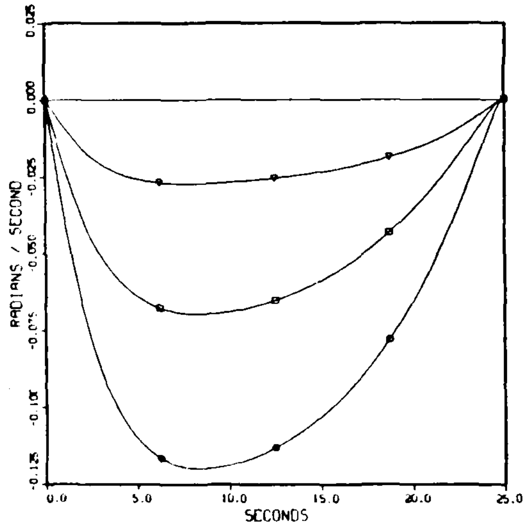
Although the evolutions of the Euler parameters do not appear to be highly sensitive to variations in inertial properties of the spacecraft, the problem is not insignificant. When the input is returned to zero at $t=t_f$ and no correction is supplied, the angular velocities are nonzero



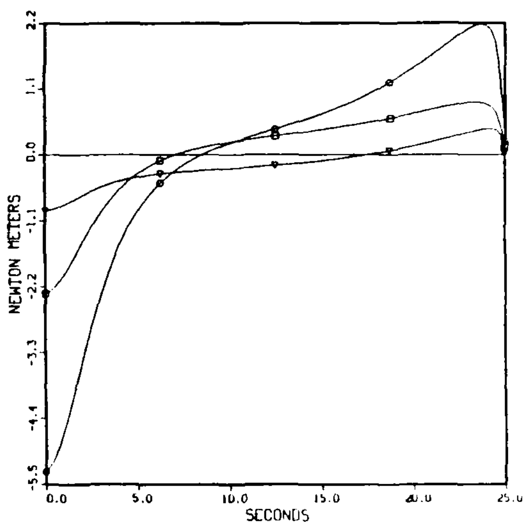
(a) Attitude Parameter (γ_o)



(b) Reduced Euler Parameters ($\underline{\gamma}$)



(c) Angular Velocity ($\underline{\omega}$)



(d) Applied Torque ($\underline{\tau}$)

Figure 4.7. State and input histories for reorientation with correction for imperfect knowledge of I'_o.
□ = 1st component, ▽ = 2nd component, ○ = 3rd component.

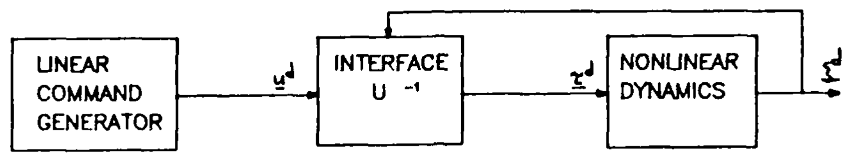
and will remain so. Not only is the desired orientation not achieved, but the result of the "rest-to-rest" maneuver is a slowly tumbling spacecraft. Correction for model variations as described gives the exact desired terminal state, and the second difficulty is overcome.

CHAPTER FIVE

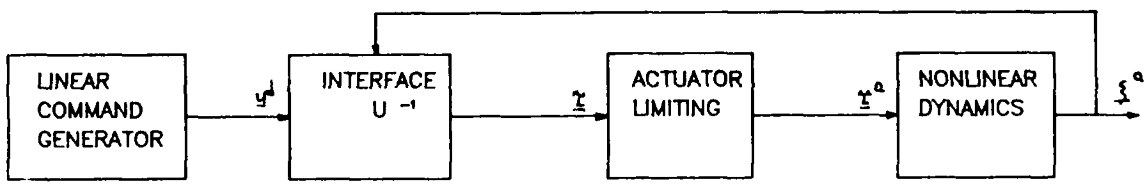
DEALING WITH DIFFICULTIES II

A general statement of the spacecraft orientation problem may be given as a requirement to move from an initial system state, $\underline{\xi}(0) = \underline{\xi}_1^o$ to within a neighborhood of the desired final state, $\underline{\xi}(t_f) = \underline{\xi}^f$, in a specified time t_f. In Chapter Three, five possible formulations of an optimal control problem were presented which can lead to a theoretical solution. However, since the problems are formulated in terms of the transformed variables, commanded torques resulting from those solutions may exceed the maximum that can be generated. Actuator saturation then occurs, and a clipped version ($\underline{\sigma}^a$) of the desired ($\underline{\sigma}^d$) torque is actually applied to the system. This situation is represented in block diagram form in Figure 5.1.

Application of other than the optimal control strategy results in a suboptimal state trajectory which may not meet the specified requirements and m.y excite the singularity $\gamma_o = 0$ in the interface between $\underline{u}$ and $\underline{\sigma}$. A method of accounting for input constraints must be developed. An associated problem is determining whether to feed the actual (sub-optimal) system state or a synthesized (optimal) state to the $\underline{u}/\underline{\sigma}$ interface. The synthesized state can be supplied by the command generator, if desired.



(a) Unconstrained torques; desired state trajectory results.



(b) Constrained torques; no compensation; perturbed state trajectory results.

Figure 5.1. Effect of constrained input torques on state trajectories.

Simplest among the possibilities of dealing with the difficulty of actuator saturation is to lobby for increased time. For a finite time optimization, this means increasing t_f. An equivalent effect is achieved in the infinite time formulation by reducing the magnitudes of the state penalties q_1 and q_2. If the time limitation is a hard constraint, then some other approach must be tried.

Typically, input constraints can be expressed as $|\sigma_i| \leq \sigma_{max}$, $i = 1,2,3$. These can be appended to the optimization problem. However, introduction of inequality constraints on the control variable significantly complicates the solution, whether the nonlinear model in terms of $(\underline{\gamma},\underline{\omega},\underline{\sigma})$ or the equivalent linear model in terms of $(\underline{x}_1,\underline{x}_2,\underline{u})$ is used. Finding a single analytic expression for the input is usually impossible.

> In solving a particular problem, constrained and unconstrained arcs must be pieced together to satisfy all the necessary conditions. At the junction points of constrained and unconstrained arcs, the control...may or may not be continuous; ... [26]

Not only is the analysis of the constrained problem more difficult, but implementation of the control strategy developed may also be impossible. Physical realization of control discontinuities can only be approximated.

A further complicating factor arises when using an equivalent linear system to determine the inputs. Bounds on $\underline{\sigma}$ in the nonlinear model do not translate to corresponding bounds on $\underline{u}$ in the linear model over the entire trajectory. To illustrate this point, a general representation of the relationships between $\underline{\sigma}$ and $\underline{u}$ is of benefit,

$$\underline{u} = \underline{h}(\underline{\xi}) + H(\underline{\xi})\underline{\sigma} = \underline{h}'(\underline{x}) + H'(\underline{x})\underline{\sigma}$$

$$\underline{\sigma} = \underline{m}(\underline{\xi}) + M(\underline{\xi})\underline{u} = \underline{m}'(\underline{x}) + M'(\underline{x})\underline{u}.$$

With $\underline{x} = T(\underline{\xi})$, the following equalities are developed.

$$\underline{h}'(\underline{x}) = \underline{h}(T(\underline{\xi})) \qquad \underline{m}'(\underline{x}) = \underline{m}(T(\underline{\xi}))$$

$$H'(\underline{x}) = H(T(\underline{\xi})) \qquad M'(\underline{x}) = M(T(\underline{\xi}))$$

$$\underline{h}(\underline{\xi}) = -(M(\underline{\xi}))^{-1}\,\underline{m}(\underline{\xi}) \qquad \underline{m}(\underline{\xi}) = -(H(\underline{\xi}))^{-1}\,\underline{h}(\underline{\xi})$$

$$H(\underline{\xi}) = (M(\underline{\xi}))^{-1} \qquad M(\underline{\xi}) = (H(\underline{\xi}))^{-1}$$

Comparing this representation to the transformation developed in the second chapter for use with external thrusters, $\underline{m}(\underline{\xi})$ and $M(\underline{\xi})$ are given as

$$\underline{m}(\underline{\xi}) = \frac{1}{2\gamma_o}(\underline{\omega}^T\underline{\omega}-4a_o)I_o\underline{\gamma} - I_o\underline{\omega}x\underline{\omega} - a_1I_o\underline{\omega}$$

$$M(\underline{\xi}) = I_o\Gamma^{-1}(\underline{\gamma})$$

and

$$\underline{\tau}(t) = \underline{m}(\underline{\xi}(t)) + M(\underline{\xi}(t))\underline{u}_1(t). \tag{21}$$

The time variable is explicitly shown here to emphasize that the dependence of $\underline{\sigma}$ on $\underline{u}$ (in this case, $\underline{\tau}$ on $\underline{u}_1$) is state related, and therefore implicitly a function of time. With M as defined, each component of $\underline{\tau}$ is dependent on all components of $\underline{u}_1$. Bounds on $\underline{\tau}$ are of the form $|\tau_i| \leq \tau_{max}$. From (21)

$$|\tau_i| = |\underline{m}_i(\underline{\xi}) + M_{i1}(\underline{\xi})u_{11} + M_{i2}(\underline{\xi})u_{12} + M_{i3}(\underline{\xi})u_{13}|$$

$$\leq |\underline{m}_i(\underline{\xi})| + |M_{i1}(\underline{\xi})u_{11}| + |M_{i2}(\underline{\xi})\underline{u}_{12}| + M_{i3}(\underline{\xi})u_{13}|.$$

If maximums could be determined for $|\underline{m}_i(\underline{\xi})|$ and $|M_{ij}(\underline{\xi})|$, j=1,2,3, then maximums could be assigned to $|u_{ij}|$, j=1,2,3, with some freedom, tc assure $|\tau_i|$ would never exceed τ_{max}. There are two drawbacks to this approach, though. First these bounds on the components of $\underline{u}_1$ could be extremely conservative, for as $\underline{\xi}$ approaches $\underline{0}$, $\underline{m}(\underline{\xi})$ also approaches $\underline{0}$ and $M(\underline{\xi})$ approaches I_o. The implication is that as the terminal state is approached, use of the full capacity of the external thrusters would be limited not by the optimization of a measure of performance, but by the artificially low constraints on the equivalent linear system input.

A second, and overriding, problem is that of finding the bounds over the entire trajectory on $m_i(\underline{\xi})$. Components of $\underline{\gamma}$ are bounded by the requirement $\gamma_o^2+\gamma_1^2+\gamma_2^2+\gamma_3^2 = 1$. However, no upper bounds on $|\omega_i|$ have been introduced. Without such constraints on $\underline{\omega}$, no constraints on $|m_i(\underline{\xi})|$ can be assumed.

Despite the complications as described, the advantages of using the equivalent linear system approach to the control problem justify trying to find a way to deal with the difficulty of actuator saturation. Three possibilities are discussed. All three presume development of an unconstrained control law. Compensation for saturation, if any, occurs when the constraint is encountered.

To provide a basis for comparison of the different approaches, a rest-to-rest maneuver is defined. An initial orientation corresponding to single-axis rotations of $5\pi/2$, $\pi/3$, and $\pi/4$ radians in a 3-1-3 convention is assumed. The objective of the maneuver is to move the

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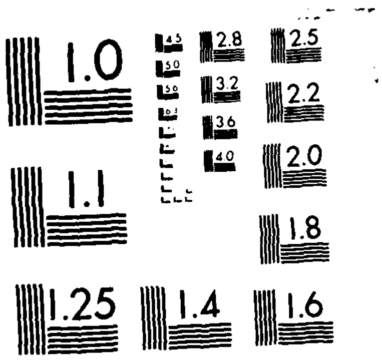

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spacecraft from the corresponding state $\underline{\gamma}(0) = (0.46180, 0.19167, 0.79987)^T$ and $\underline{\omega}(0) = \underline{0}$ to the terminal state $\underline{\gamma}(t_f) = \underline{0}$, $\underline{\omega}(t_f) = \underline{0}$. Requirements which must be met for a result to be considered satisfactory are $|\gamma_i(t_f)| \leq 0.01$, $|\omega_i(t_f)| \leq 0.005$ rad/sec, i=1,2,3 with t_f=30 seconds a hard constraint. External thrusters will be used to accomplish the task with actuator constraints $|\tau_i| \leq 2.0$ Nm. The inertial matrix I_o is unchanged from the previous chapters. Any of the five performance measures discussed in Chapter Three may be used with penalties chosen to allow the unconstrained problem to satisfy the terminal conditions. State trajectories and input histories for the unconstrained problem are shown in Figures 5.2 through 5.6.

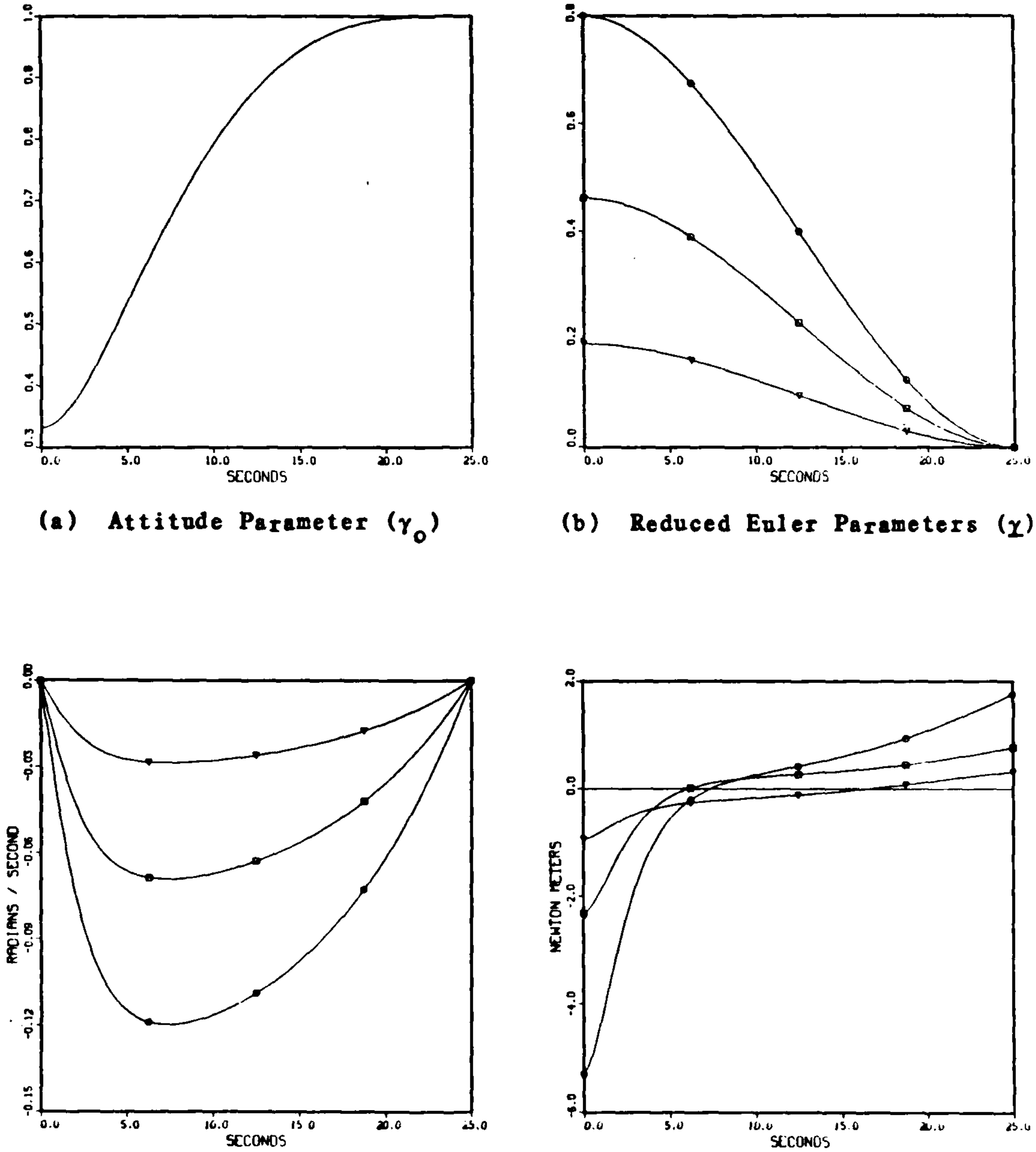
5.1 Uncompensated Saturation

If the reality of input constraints is either unanticipated or knowingly ignored, a policy of uncompensated saturation is implicitly defined. This policy is described in block diagram form in Figure 5.1. Actuators perform at their limits when the commanded torque is greater in magnitude than τ_{max}. No compensation is introduced to correct for the suboptimal torque. Mathematically,

$$(\underline{\tau}^a)_i = \begin{bmatrix} (\underline{\tau}^d)_i = [\underline{m}(\underline{\xi}) + M(\underline{\xi})\underline{u}]_i & \text{if} \quad |(\underline{\tau}^d)_i| \leq \tau_{max} \\ [\text{sgn}\ (\underline{\tau}^d)_i]\tau_{max} & \text{if} \quad |(\underline{\tau}^d)_i| > \tau_{max} \end{bmatrix}$$

for i=1,2,3.

The resulting state trajectory will correspond to the desired state trajectory only if the desired torques never exceed the actuator limits.



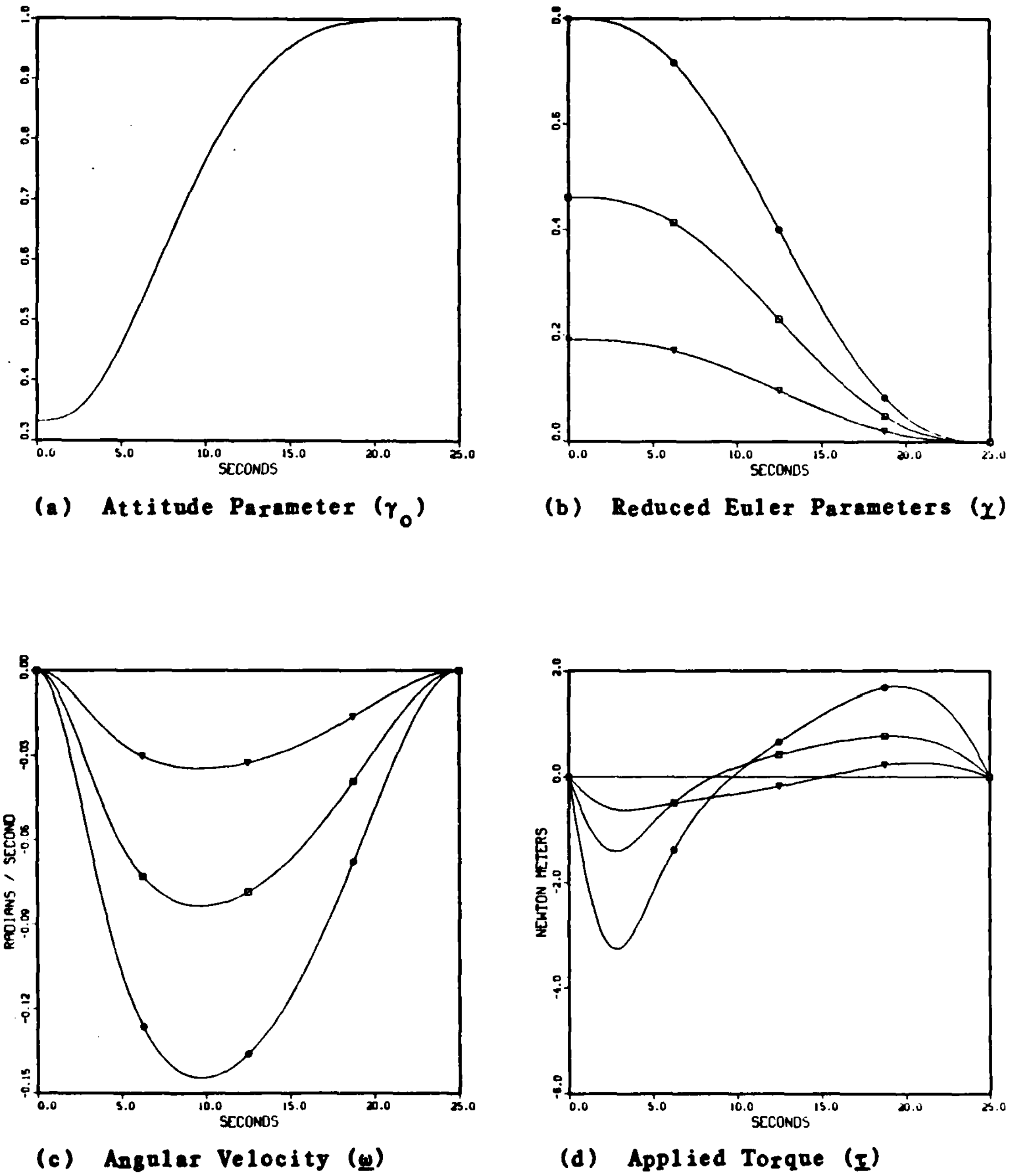
(a) Attitude Parameter (γ_o)

(b) Reduced Euler Parameters ($\underline{\gamma}$)

(c) Angular Velocity ($\underline{\omega}$)

(d) Applied Torque ($\underline{\tau}$)

Figure 5.2. Reorientation state and input histories minimizing J_1 with unconstrained torques.
□ = 1st component, ▽ = 2nd component, ○ = 3rd component.



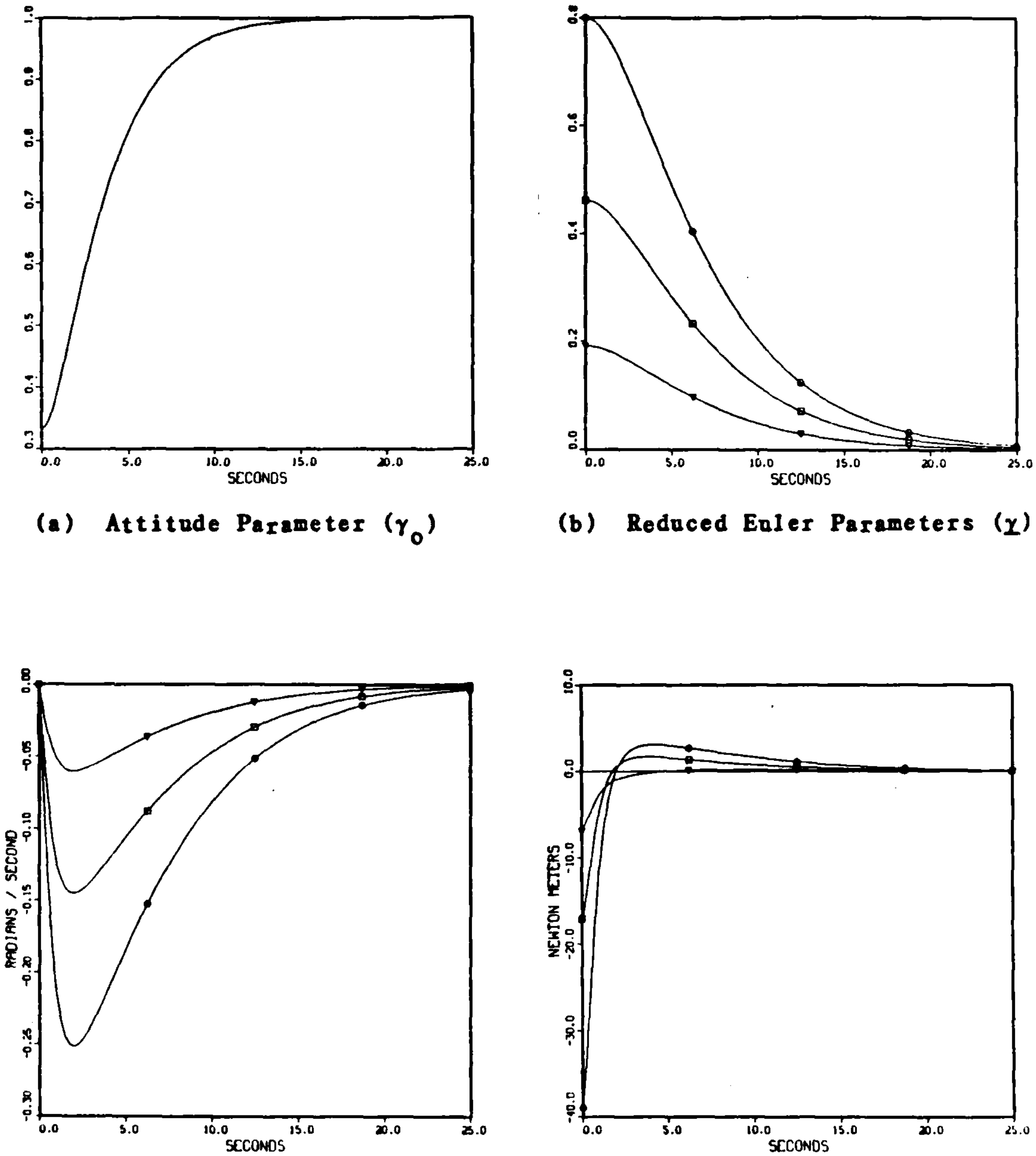
(a) Attitude Parameter (γ_0)

(b) Reduced Euler Parameters ($\underline{\gamma}$)

(c) Angular Velocity ($\underline{\omega}$)

(d) Applied Torque ($\underline{\tau}$)

Figure 5.3. Reorientation state and input histories minimizng J_2 with unconstrained torques.
□ = 1st component, ▽ = 2nd component, ○ = 3rd component.



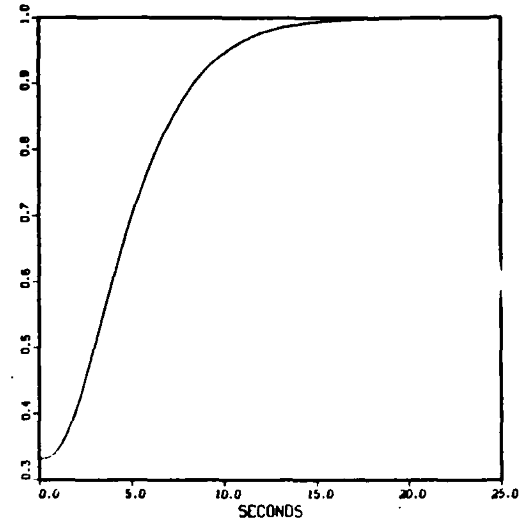
(a) Attitude Parameter (γ_o)

(b) Reduced Euler Parameters ($\underline{\gamma}$)

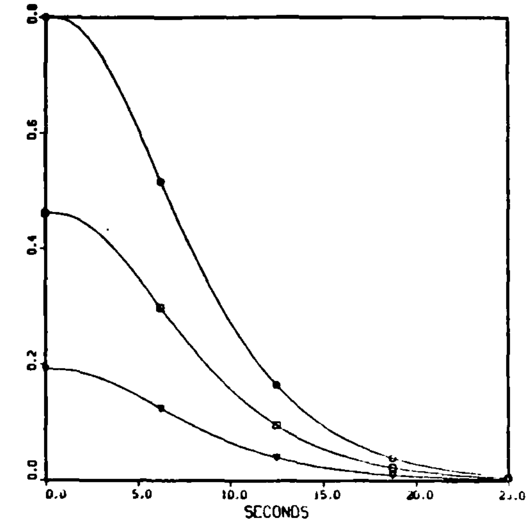
(c) Angular Velocity ($\underline{\omega}$)

(d) Applied Torque ($\underline{\tau}$)

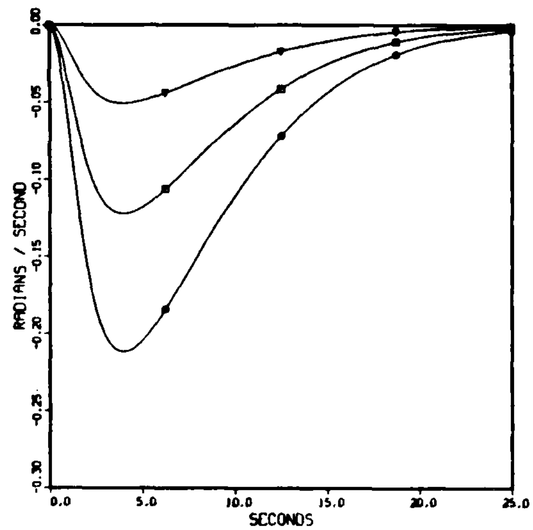
Figure 5.4. Reorientation state and input histories minimizing J_3 with unconstrained torques.
□ = 1st component, ▽ = 2nd component, ○ = 3rd component.



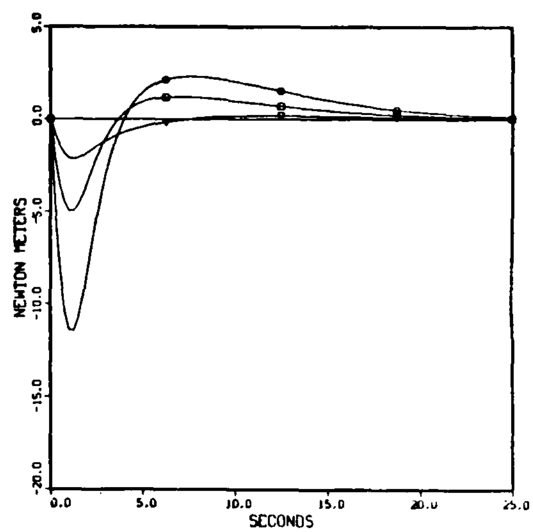
(a) Attitude Parameter (γ_o)



(b) Reduced Euler Parameters ($\underline{\gamma}$)

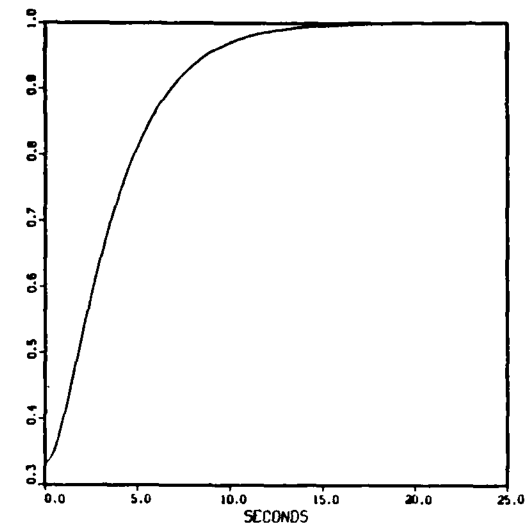


(c) Angular Velocity ($\underline{\omega}$)

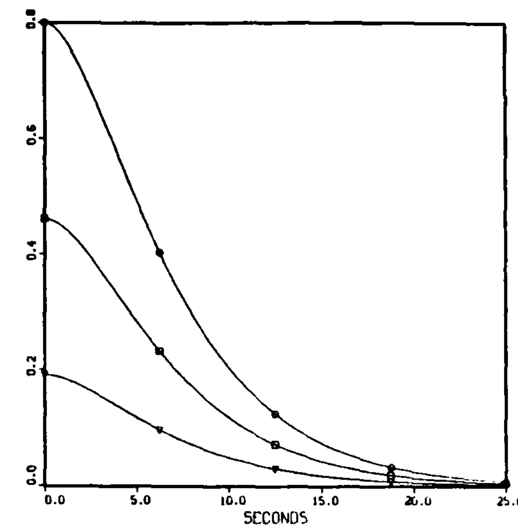


(d) Applied Torque ($\underline{\tau}$)

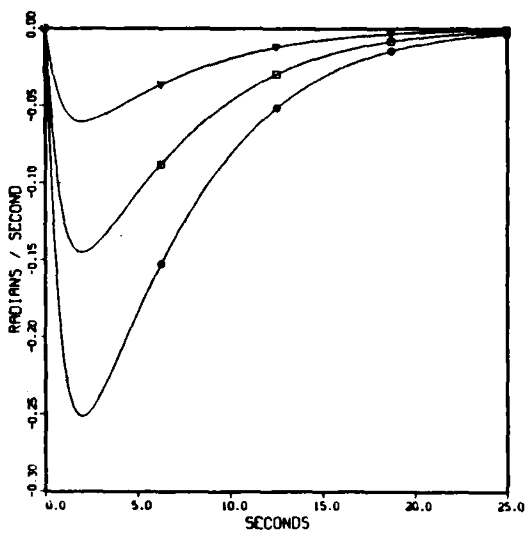
Figure 5.5. Reorientation state and input histories minimizing J_4 with unconstrained torques.
□ = 1st component, ▽ = 2nd component, ○ = 3rd component.



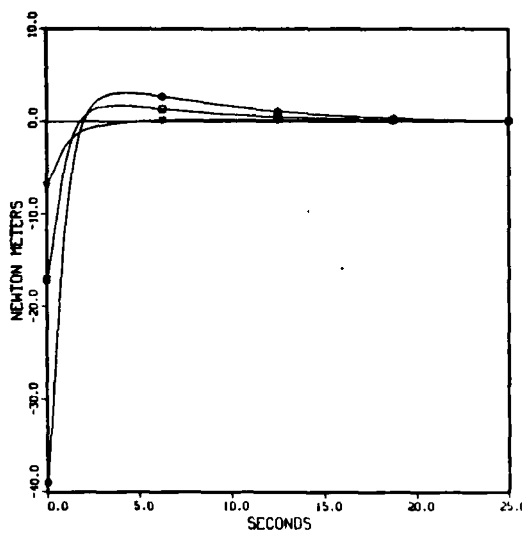
(a) Attitude Parameter (γ_0)



(b) Reduced Euler Parameters ($\underline{\gamma}$)



(c) Angular Velocity ($\underline{\omega}$)



(d) Applied Torque ($\underline{\tau}$)

Figure 5.6. Reorientation state and input histories for the damped system with unstrained torques.
□ = 1st component, ▽ = 2nd component, ○ = 3rd component.

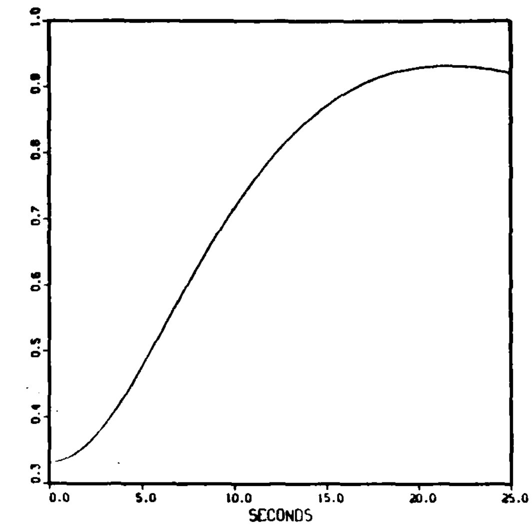
When open loop control is used with no correction for saturation, the error will be cumulative throughout the task. If feedback control is desired, the infinite time optimization must be used to avoid infinite terminal gains. Figures 5.4, 5.5 and 5.6 show that for the infinite time problems, τ_{max} is exceeded by the commanded torque more in terms of both time and magnitude than for the finite time scenarios.

The expectation that trajectories generated without some compensation for input constraints will not satisfy the requirements is confirmed by the simulation results shown in Figures 5.7 through 5.10. Infinite time problem formulation leads to excitation of the input transformation singularity when the equivalent linear system is a series of integrators. However, introduction of damping results in such significant improvement that the specifications are nearly met (Figure 5.11).

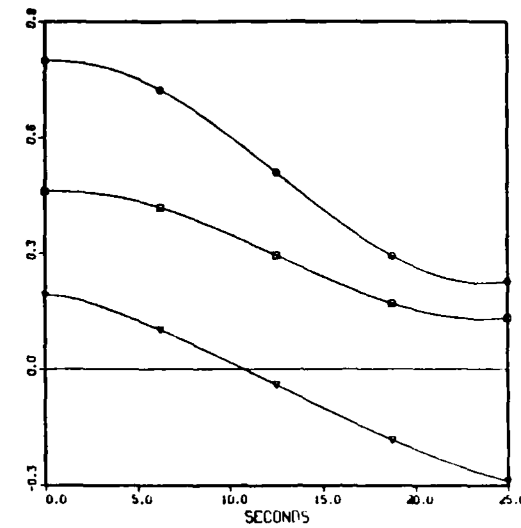
5.2 Optimal Decision Strategy

Use of a pointwise torque optimization scheme is suggested by Spong et al. [27] to compensate for bounded input in the control of robot manipulators. In terms of the robotic application, the objective of the optimal decision strategy is to minimize a weighted norm of the difference between achieved and desired joint acceleration vectors at each sample time along the trajectory. Statement of the corresponding objective for spacecraft applications requires only replacing "joint acceleration vectors" by "angular acceleration vectors."

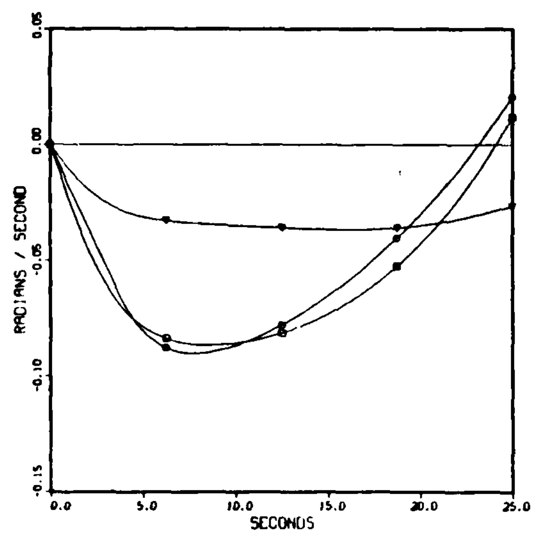
A reference trajectory for the nonlinear system model, $\underline{\xi}^d$, is required for this approach. Transformation of the solution of the unconstrained problem conveniently provides this reference trajectory. The optimal decision strategy is then formulated at each sample time,



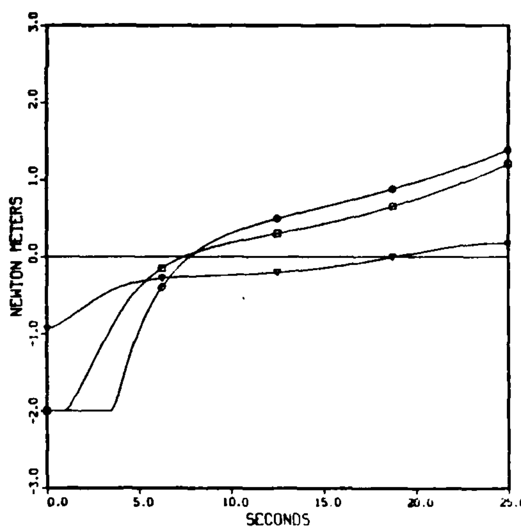
(a) Attitude Parameter (γ_o)



(b) Reduced Euler Parameters ($\underline{\gamma}$)

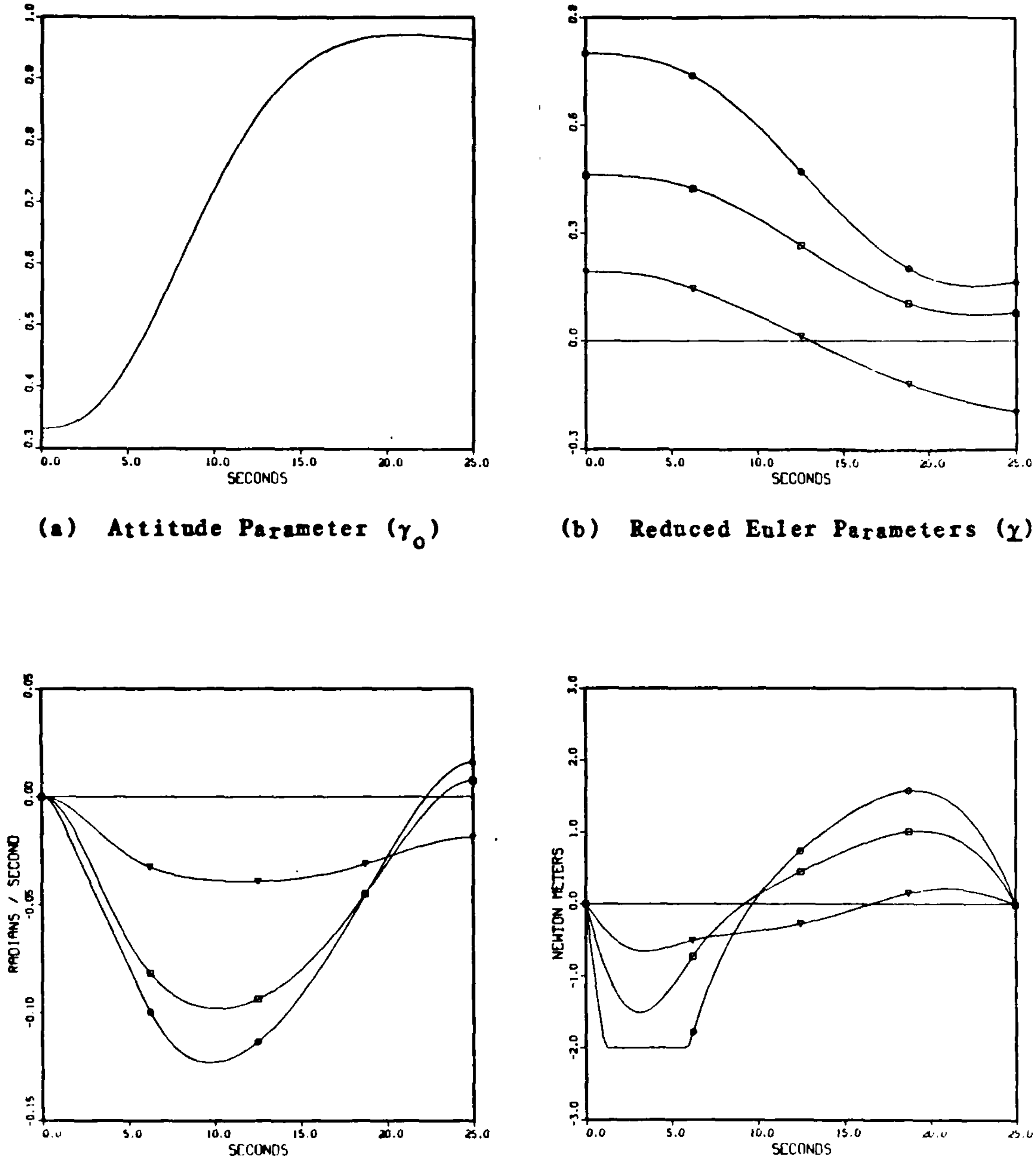


(c) Angular Velocity ($\underline{\omega}$)



(d) Applied Torque ($\underline{\tau}$)

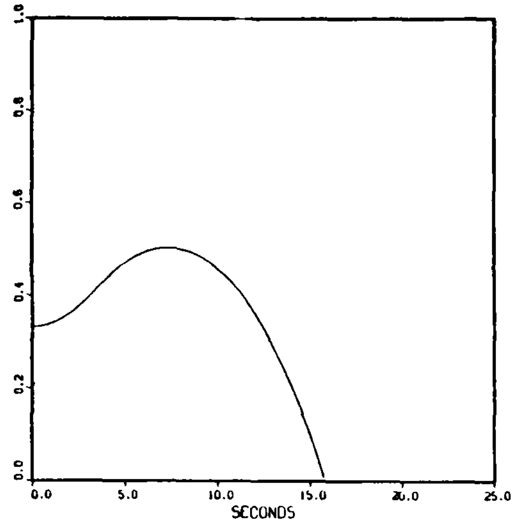
Figure 5.7. Reorientation state and input histories minimizing J_1 uncompensated for constrained torques.
□ = 1st component, ▽ = 2nd component, ○ = 3rd component.



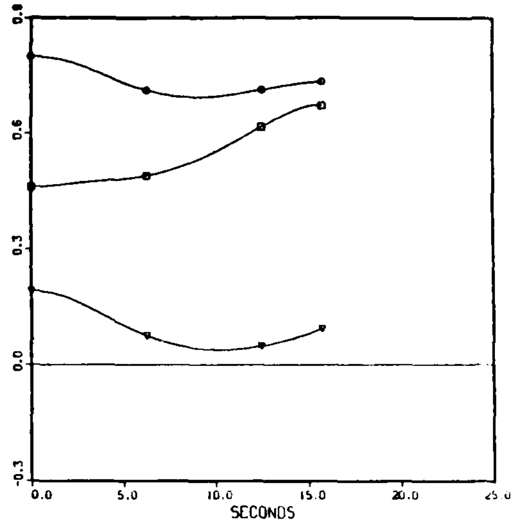
(a) Attitude Parameter (γ_0) **(b) Reduced Euler Parameters ($\underline{\gamma}$)**

(c) Angular Velocity ($\underline{\omega}$) **(d) Applied Torque ($\underline{\tau}$)**

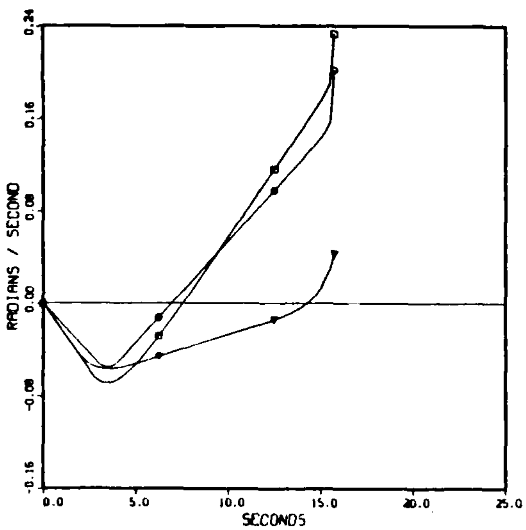
Figure 5.8. Reorientation state and input histories minimizing J_2 uncompensated for constrained torques.
□ = 1st component, ▽ = 2nd component, ○ = 3rd component.



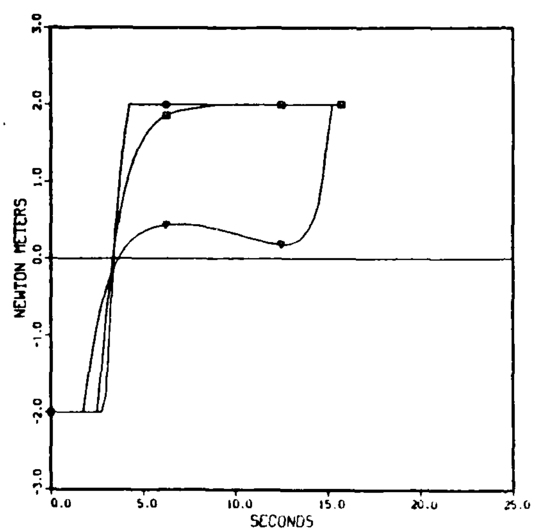
(a) Attitude Parameter (γ_o)



(b) Reduced Euler Parameters ($\underline{\gamma}$)

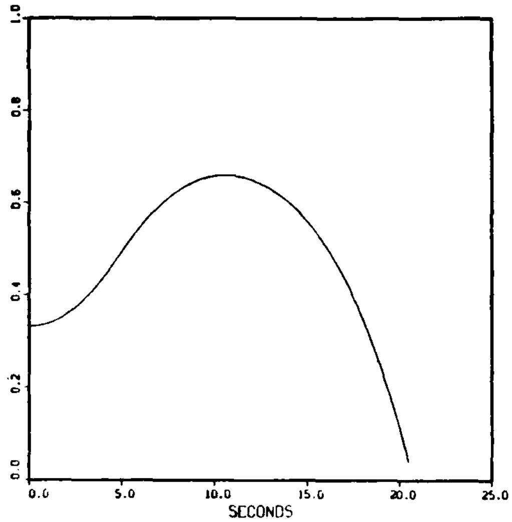


(c) Angular Velocity ($\underline{\omega}$)

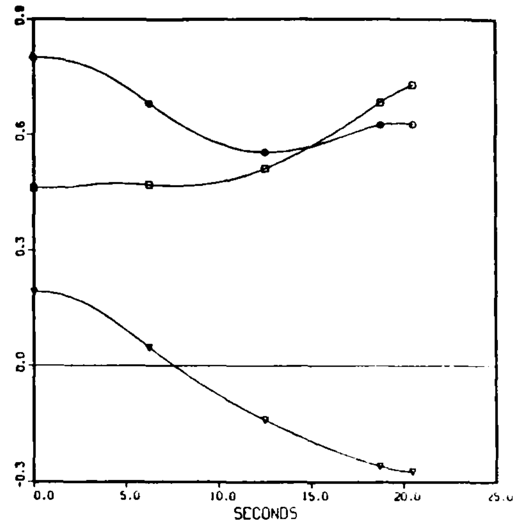


(d) Applied Torque ($\underline{\tau}$)

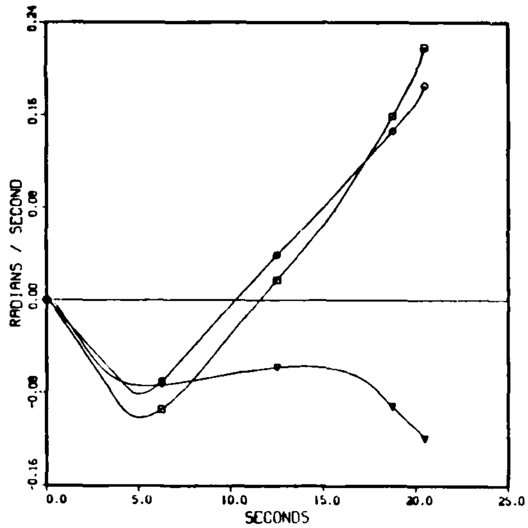
Figure 5.9. Reorientation state and input histories minimizing J_3 uncompensated for constrained torques.
□ = 1st component, ∇ = 2nd component, ○ = 3rd component.



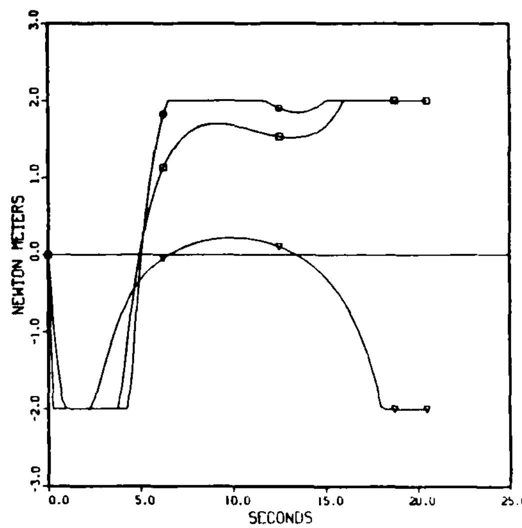
(a) Attitude Parameter (γ_o)



(b) Reduced Euler Parameters ($\underline{\gamma}$)

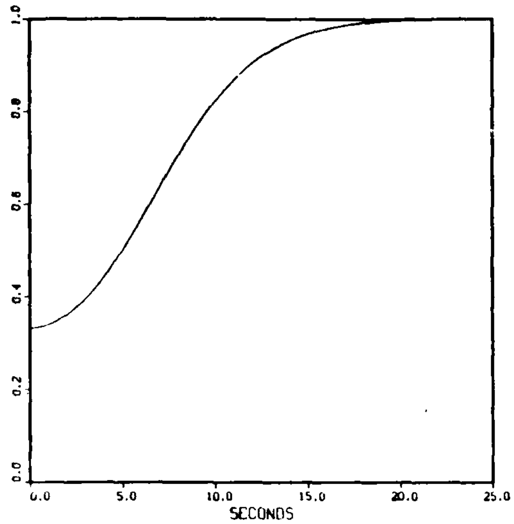


(c) Angular Velocity ($\underline{\omega}$)

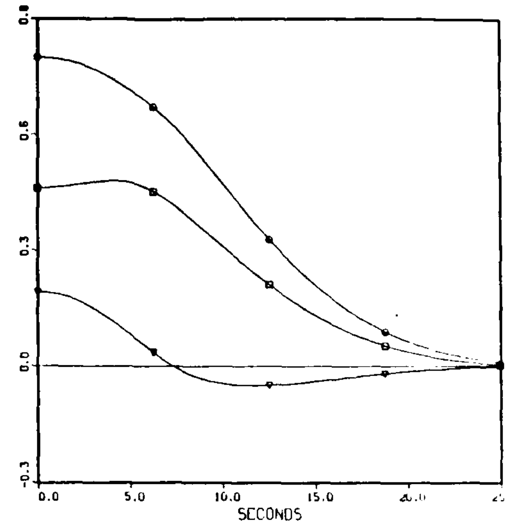


(d) Applied Torque ($\underline{\tau}$)

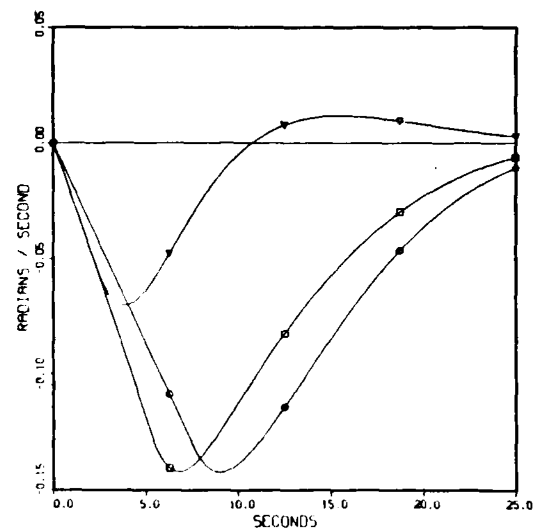
Figure 5.10. Reorientation state and input histories minimizing J_4 uncompensated for constrained torques.
□ = 1st component, ▽ = 2nd component, ○ = 3rd component.



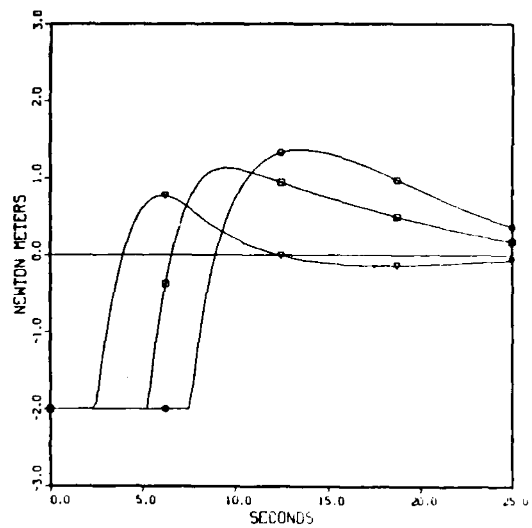
(a) Attitude Parameter (γ_o)



(b) Reduced Euler Parameters ($\underline{\gamma}$)



(c) Angular Velocity ($\underline{\omega}$)



(d) Applied Torque ($\underline{\tau}$)

Figure 5.11. Reorientation state and input histories for the damped system uncompensated for constrained torques. □ = 1st component, ▽ = 2nd component, ○ = 3rd component.

and bounds determined for the maneuver time which will guarantee satisfaction of the inequality. The latter approach is computationally cleaner and easier to implement.

Excitation of the singularity $\gamma_o = 0$, equivalent to $||\chi|| = 1$, in the inverse transformation does not occur in rest-to-rest maneuvers except where $||\chi(0)|| = 1$. In such an instance, division of the problem into two reorientation sub-tasks, with appropriate choice of inertial frame for each task, circumvents the singularity. Solution of the unconstrained problem when the spacecraft is initially tumbling can lead to a violation of $||\chi(t)|| < 1$. A shorter maneuver time $t_1 < t_f$ can be found which will avoid the violation. Bounds for this time are dependent on $||\mathbf{x}_1^o||$, $||\mathbf{x}_2^o||$, and the control strategy chosen. Another approach to singularity avoidance is to divide the tumbling-to-orientation problem into a detumbling task followed by a rest-to-rest maneuver. Inertial frames may again be judiciously chosen for each task. Step inputs to the linear model result in a more liberal bound on the time to detumble, and allow the maneuver to be accomplished with lower torque magnitudes.

Imperfect knowledge of the spacecraft inertial properties leads to a perturbed state trajectory and nonzero terminal angular velocity. A control which was designed to reorient the system to a static desired attitude in fact leaves the spacecraft tumbling slowly. Introduction of a linear regulator acting on the state error corrects for this perturbation and allows the desired terminal state to be achieved.

Torque input constraints are difficult to deal witn. Attempts to append these constraints to the optimization problem eliminate the benefits of using an equivalent linear system for control design. As

$$\underline{\sigma} = \left[\frac{\partial \phi_1}{\partial \xi_2} \, \psi\right]^{-1} \cdot \left[\underline{u} - a_1\phi_1 - a_o\xi_1 - \frac{\partial \phi_1}{\partial \xi_1} \phi_1 - \frac{\partial \phi_1}{\partial \xi_2} \phi_2\right] .$$

Spacecraft attitude control, a highly nonlinear problem in the physical realm, is greatly simplified by this linearizing transformation approach.

A number of immediate advantages over previous control design techniques are realized. Use of analytical tools for linear systems with the equivalent model allows design and implementation of fast, multi-axial large-angle maneuvers. Convergence of a series of solutions is not an issue; exact solutions are achieved. Subsequent maneuvers of the same type do not require resolving the problem. The new control is given by substituting in the original solution the new initial time, initial conditions, and final time. Exact terminal control may be realized by appropriate choice of performance criteria in the linear space.

For the spacecraft attitude control problem, the inverse $\left[\frac{\partial \phi_1}{\partial \xi_2} \, \psi\right]^{-1} = \left[\Gamma(\mathbf{x}_1) I_o^{-1}\right]^{-1}$ fails to exist on the subspace defined by $||\mathbf{x}_1|| = 1$. Since the geometrically imposed identity on the Euler parameters forces $\gamma_0^2+\gamma_1^2+\gamma_2^2+\gamma_3^2 = 1$, control strategies must be developed which result in state trajectories for which $||\mathbf{x}_1(t)|| < 1$. This restriction may be imposed directly as a nonlinear state constraint in the optimization problem used to derive the control. Use of the full set of Euler parameters would not lead to singularities. However, the state constraint would become $||\mathbf{x}_1(t)|| = 1$, which is intractable as a linear problem. Alternatively, an unconstrained problem may be solved

CHAPTER SIX

CONCLUSIONS AND RECOMMENDATIONS FOR FURTHER STUDY

6.1 Conclusions

Results of the work described here can be applied to problems other than spacecraft attitude control. Many physical processes are modeled by a vector second order system

$$\dot{\underline{\xi}}_1 = \underline{\phi}_1(\underline{\xi})$$

$$\dot{\underline{\xi}}_2 = \underline{\phi}_2(\underline{\xi}) + \Psi(\underline{\xi})\underline{\sigma}$$

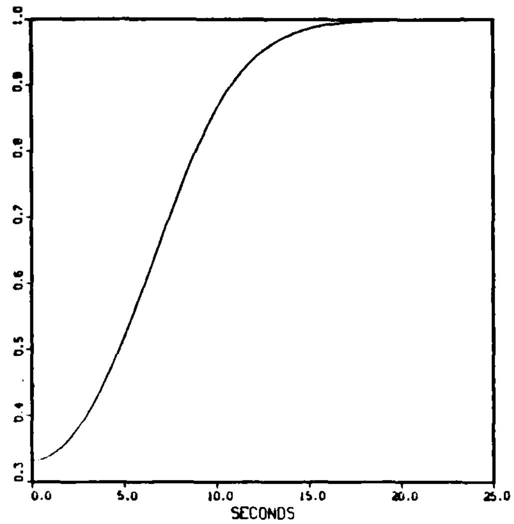
where $\underline{\xi} = (\underline{\xi}_1^T, \underline{\xi}_2^T)^T$. For systems of this type, a general vector second order equivalent linear model can be defined,

$$\dot{\underline{x}}_1 = \underline{x}_2$$

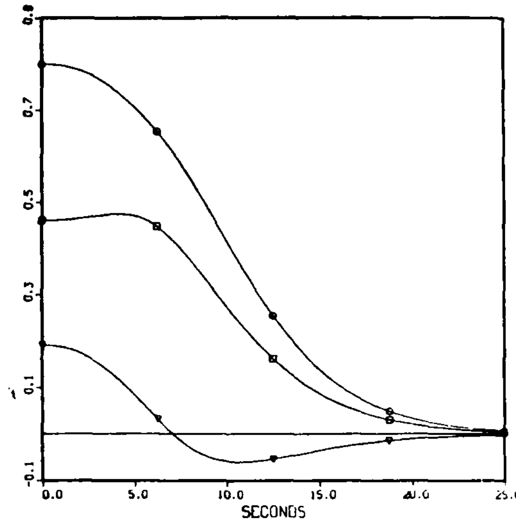
$$\dot{\underline{x}}_2 = -a_o\underline{x}_1 - a_1\underline{x}_2 + \underline{u}$$

where $\underline{x}_1 = \underline{\xi}_1$, $\underline{x}_2 = \dot{\underline{\xi}}_1$, and $\underline{u} = \ddot{\underline{\xi}}_1 + a_1\dot{\underline{\xi}}_1 + a_o\underline{\xi}_1$. The usefulness of this equivalent linear model is determined by the nature of $\left[\frac{\partial(\underline{\phi}_1(\underline{\xi}))}{\partial\underline{\xi}_2}\Psi(\underline{\xi})\right]^{-1}$, for where this inverse exists, the physical input is recovered from the equivalent linear input by

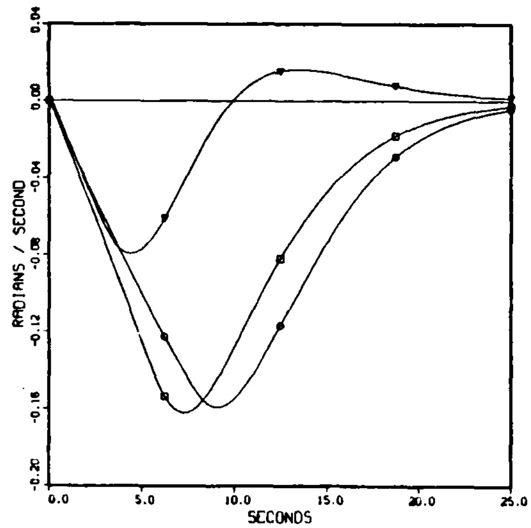
An important point to emphasize is that reinitialization will not always produce the desired results. A finite amount of time is required after the last commanded torque saturation to allow the reinitialized control strategy to compensate for that limiting. Even then, task requirements can be set so stringent that the constrained problem cannot be implemented. The reinitialization approach to dealing with input constraints is proposed as one tool with which exact terminal control may be realized.



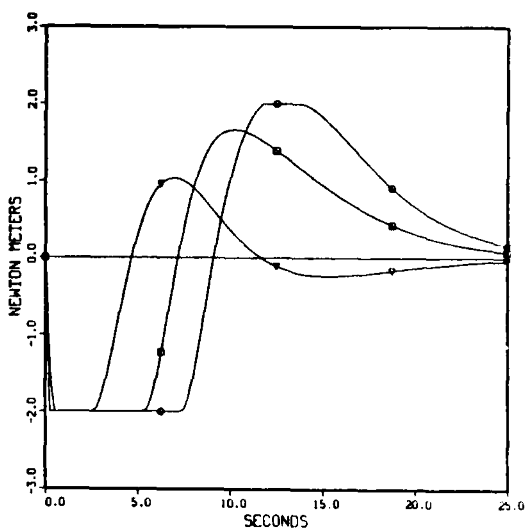
(a) **Attitude Parameter (γ_0)**



(b) **Reduced Euler Parameters ($\underline{\gamma}$)**

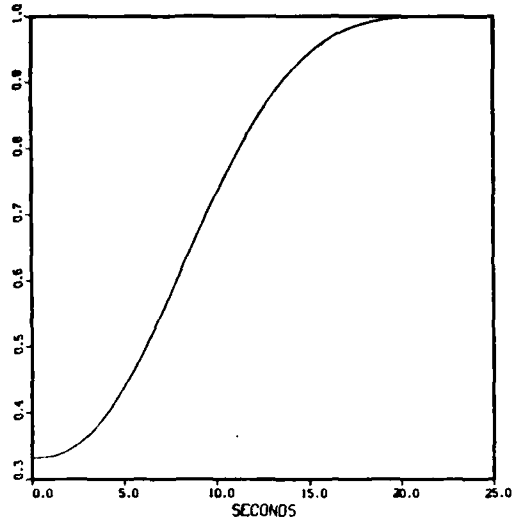


(c) **Angular Velocity ($\underline{\omega}$)**

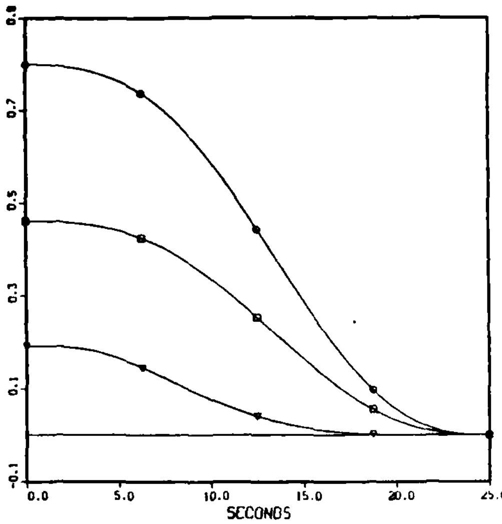


(d) **Applied Torque ($\underline{\tau}$)**

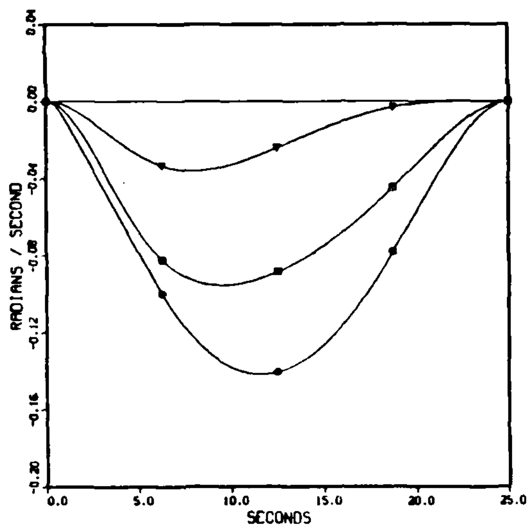
Figure 5.18. Reorientation state and input histories minimizing J_4 with reinitialization after torque saturation. □ = 1st component, ▽ = 2nd component, ○ = 3rd component.



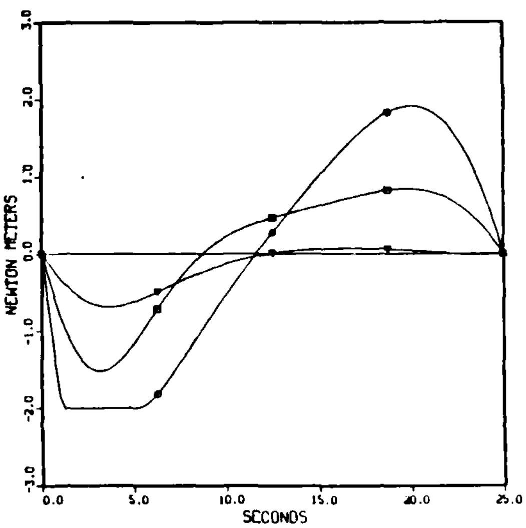
(a) Attitude Parameter (γ_0)



(b) Reduced Euler Parameters ($\underline{\gamma}$)



(c) Angular Velocity ($\underline{\omega}$)



(d) Applied Torque ($\underline{\tau}$)

Figure 5.17. Reorientation state and input histories minimizing J_2 with reinitialization after torque saturation.
□ = 1st component, ▽ = 2nd component, ○ = 3rd component.

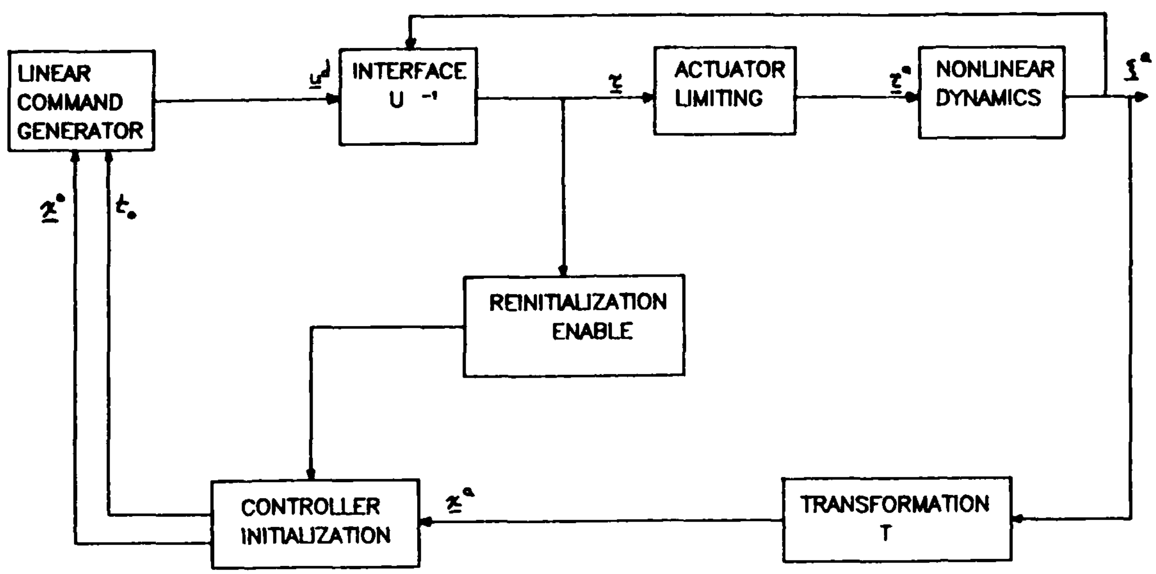


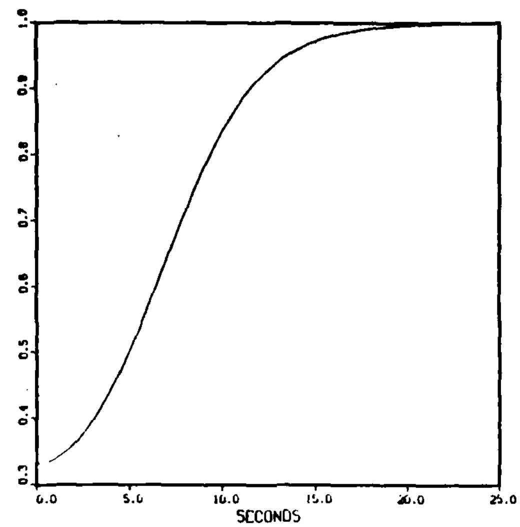
Figure 5.18. Correction for constrained input torques by Reinitialization of Optimization Problem.

which are the desired or limited values as appropriate, the current states are transformed to linear model equivalents $\underline{x}_1(t_k)$, $\underline{x}_2(t_k)$, and $\underline{u}(t_k)$. The command generator is then updated (reinitialized) using the current sample time t_k as t_o and the equivalent linear states and input as initial conditions. Final time t_f remains unchanged throughout the maneuver. When the commanded torque no longer exceeds the physical limits in any component, updating of the parameters in $\underline{u}(t)$ ceases until saturation is again encountered. Effectively, the optimization problem has been reinitialized at a time when the unconstrained problem is applicable. Figure 5.16 pictorially presents this approach.

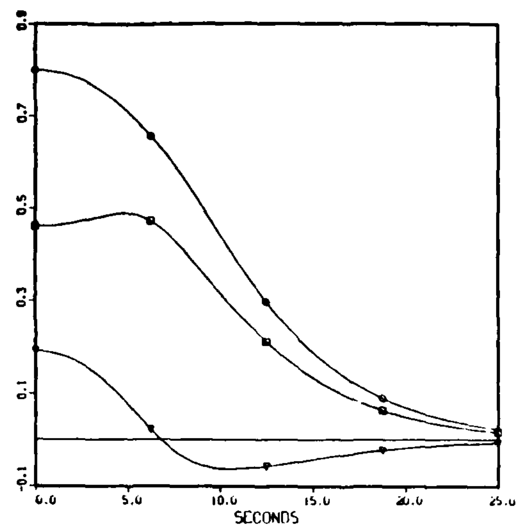
The reinitialization of the optimization problem is only applied in the context of Cases II and IV. Penalizing $||\dot{\underline{u}}||$ allows matching of the torque vector at times t_k and t_{k+1}. Discontinuities in torque profiles are thereby avoided. Simulated results for these two cases are shown in Figures 5.17 and 5.18.

5.4 Discussion

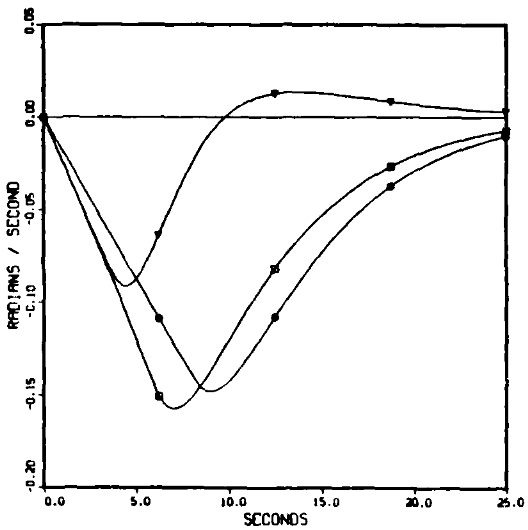
Disregard of torque actuator limits can have catastrophic results in spacecraft attitude control as shown in the trajectories of Section 5.1. Use of a damped second-order equivalent linear system significantly improves the uncompensated performance, but fails to satisfy terminal requirements. Applying an optimal decision strategy in a path-following scenario gives good results for infinite time control. However, the exact desired terminal state is not achieved. By reinitializing the solution of the optimal control problem at times throughout the maneuver where commanded torques exceed actuator limits, exact terminal control can again be achieved in finite time.



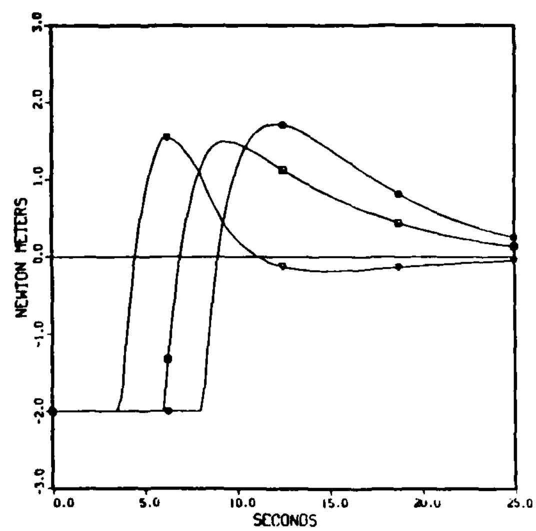
(a) Attitude Parameter (γ_o)



(b) Reduced Euler Parameters ($\underline{\gamma}$)

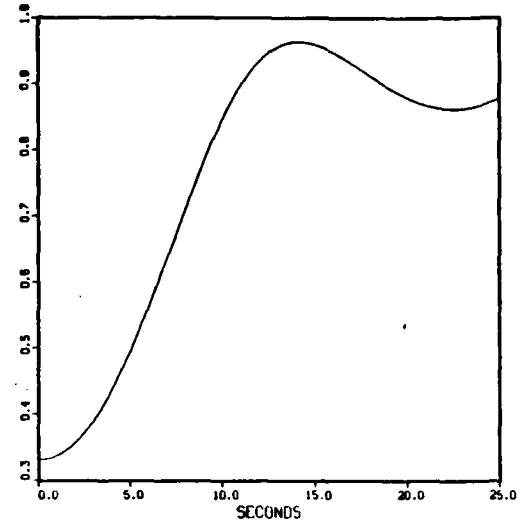


(c) Angular Velocity ($\underline{\omega}$)

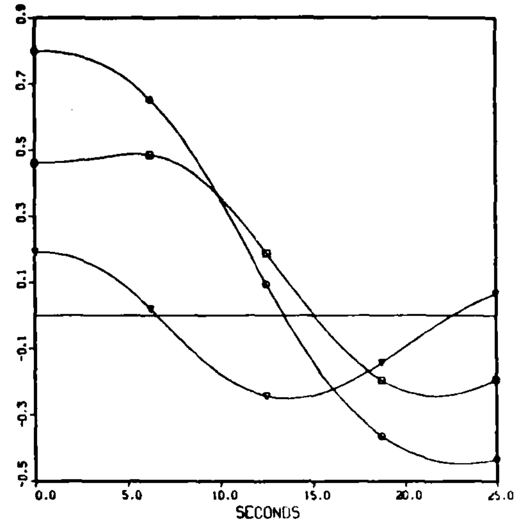


(d) Applied Torque ($\underline{\tau}$)

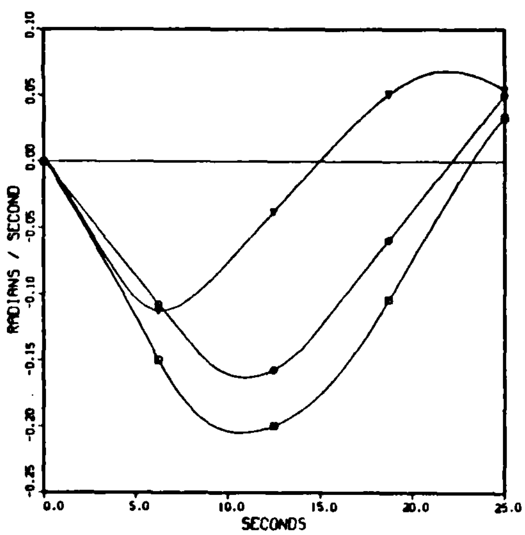
Figure 5.15. Reorientation state and input histories for the damped system with ODS compensation for constrained torques. □ = 1st component, ▽ = 2nd component, ○ = 3rd component.



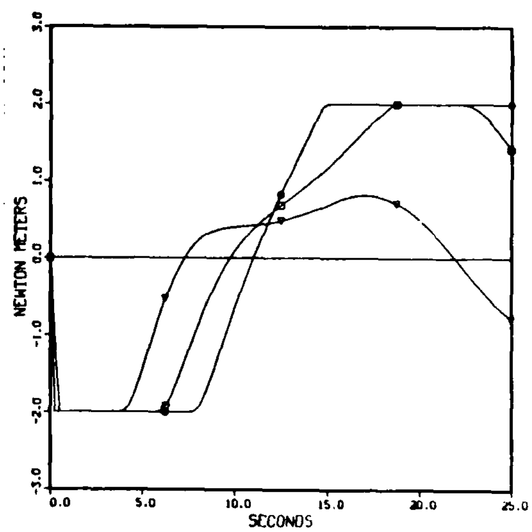
(a) Attitude Parameter (γ_o)



(b) Reduced Euler Parameters ($\underline{\gamma}$)

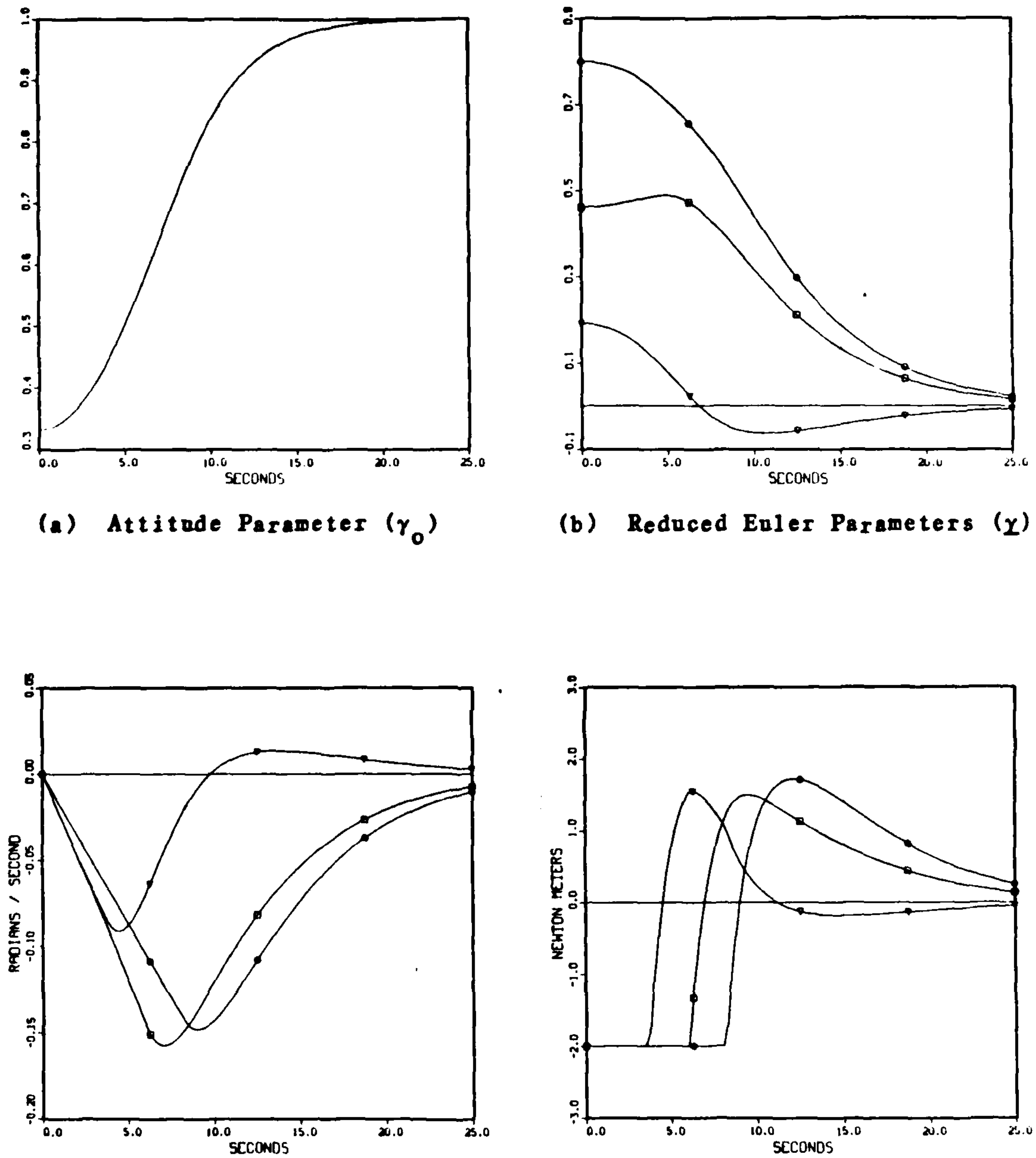


(c) Angular Velocity ($\underline{\omega}$)



(d) Applied Torque ($\underline{\tau}$)

Figure 5.14. Reorientation state and input histories minimizing J_4 with ODS compensation for constrained torques.
□ = 1st component, ▽ = 2nd component, ○ = 3rd component.



(a) Attitude Parameter (γ_o)

(b) Reduced Euler Parameters ($\underline{\gamma}$)

(c) Angular Velocity ($\underline{\omega}$)

(d) Applied Torque ($\underline{\tau}$)

Figure 5.13. Reorientation state and input histories minimizing J_3 with ODS compensation for constrained torques.
□ = 1st component, ▽ = 2nd component, ○ = 3rd component.

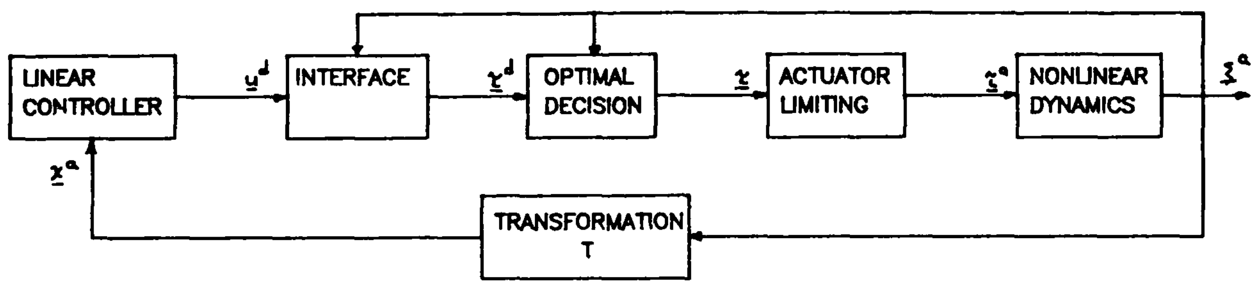


Figure 5.12. Correction for constrained torques by Optimal Decision Strategy.

"optimum" constrained $\underline{\tau}$, but the code must be fast enough to converge between samples for real time implementation.

The "Spong ODS" technique for dealing with bounded inputs is formulated in the context of infinite time control using state feedback. Figure 5.12 is a block diagram representation of this approach. Simulations shown in Figures 5.13 through 5.15 correspond to cases III, IV, and V in Chapter Three.

5.3 Reinitialization of Optimization Problem

An alternative approach to compensating for actuator saturation makes use of the principle of optimality. A control policy optimal over an interval $[0,t_f]$ is optimal over all subintervals $[t_o,t_f]$ [29]. Closed form expressions given in Chapter Three for the control policies depend on the parameters initial time t_o, final time t_f, initial states $\underline{x}_1^o$ and $\underline{x}_2^o$, and for Cases II and IV, initial linear model acceleration $\underline{u}^o$. (Expressions for arbitrary initial time t_o are given in Appendix B.) At any time during a maneuver, the current time, current linear model state, and, when appropriate, current linear model acceleration can be introduced into the optimal control expression, and the resulting trajectory is optimal in the same sense as before.

This concept is applied to the spacecraft attitude control problem to develop a constrained control policy. The unconstrained problem is solved initially, using a performance measure that penalizes $||\dot{\underline{u}}||$. When any commanded torque exceeds the input constraint, that actuator is operated at its limit. At each sample time t_k during saturated operation of one or more of the actuators, the states $\underline{\gamma}(t_k)$ and $\underline{\omega}(t_k)$ are measured. Along with the torque vector $\underline{\tau}(t_k)$, the components of

$$\text{minimize} \quad \frac{1}{2}\,(\dot{\underline{\omega}} - \dot{\underline{\omega}}^d)^T\, S\, (\dot{\underline{\omega}} - \dot{\underline{\omega}}^d) \tag{22}$$

$$\text{subject to} \qquad |\tau_i| \leq \tau_{max}, \qquad i=1,2,3$$

where the weighting matrix S is symmetric and positive definite, but otherwise freely chosen to improve system performance. (Unfortunately, no guidelines are suggested for choosing S. In the simulation results of the referenced paper [27], the authors imply the weighting matrix is determined by trial and error.)

Substituting the known expressions $\dot{\underline{\omega}} = I_o^{-1}(I_o\underline{\omega}x\underline{\omega}) + I_o^{-1}\underline{\tau}$ and $\dot{\underline{\omega}}^d = I_o^{-1}(I_o\underline{\omega}^d x\underline{\omega}^d) + I_o^{-1}\underline{\tau}^d$ in (22) gives an objective function in terms of the free variable $\underline{\tau}$. At any instant of time, t_k, all other parameters $(I_o, \underline{\omega}(t_k), \underline{\omega}^d(t_k),\ \underline{\tau}^d(t_k))$ are fixed. Factoring out I_o^{-1} and rearranging terms, the problem statement becomes

$$\text{minimize} \quad \frac{1}{2}\,(\underline{\tau}+\underline{v}(t_k))^T I_o^{-T} S I_o^{-1} (\underline{\tau}+\underline{v}(t_k)) \tag{23}$$

$$\text{subject to} \qquad |\tau_i| \leq \tau_{max} \qquad i=1,2,3$$

where $\underline{v} = I_o\underline{\omega}(t_k)x\underline{\omega}(t_k) - I_o\underline{\omega}^d(t_k)x\underline{\omega}^d(t_k) - \underline{\tau}^d(t_k)$. Since I_o is a symmetric positive definite matrix, the product $I_o^{-T}SI_o^{-1}$ is also symmetric positive definite, and the objective function (23) is convex. With a convex objective function, satisfaction of the Kuhn Tucker conditions is sufficient to solve the minimization problem [28]. Any of a number of quadratic programming methods may be used to find the

with the state constraints, an easier approach is to solve the unconstrained problem and to compensate after the fact for commanded torques in excess of the constraint. An optimal decision strategy minimizes at each sample time a quadratic function of the difference between desired and achievable angular acceleration. Improvement is realized over trajectories for which no compensation for actuator saturation is implemented. However, if exact terminal control is required, this method of correction falls short. A technique which reinitializes the optimal control problem for the linear model whenever saturation is encountered is introduced here. With this approach, the exact terminal state may be achieved.

In spite of the sophisticated spacecraft control algorithms described herein and available elsewhere, mission specifications can be developed which none of the techniques will satisfy. Open lines of communication between the mission planner and the control engineer are still essential.

6.2 Recommendations for Further Study

Mathematical descriptions of upper bounds on maneuver times for avoiding the input transformation singularity are extremely conservative. This is evidenced by the example in Chapter Four where the bound was 10.83 seconds but the task could last 45 seconds without violating the state constraint. More liberal maneuver time bounds need to be defined. A possible approach may be to form $||\mathbf{x}_1(t)||^2 = \mathbf{x}_1^T(t)\mathbf{x}_1(t)$ and find the maximum as a function of $||\mathbf{x}_1^o||$, $||\mathbf{x}_2^o||$, and t_f. Applying the inequality $||\mathbf{x}_1|| < 1$ to that point in the trajectory may lead to better estimates of allowable mission durations.

A single closed form solution to the optimization problems which incorporates the state ($||\underline{x}_1|| < 1$) and input ($|\sigma_i| \leq \sigma_{max}$) constraints would eliminate the requirement for determining bounds on maneuver time or compensation for actuator saturation. The state constraint can be expressed as $x_{11}^2+x_{12}^2+x_{13}^2 < 1$, and the input constraint can be stated $N\underline{\sigma} \leq \underline{c}$ where

$$N^T = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix} \qquad \text{and} \qquad \underline{c}^T = \sigma_{max}\ (1,1,1,1,1,1).$$

A state and input constrained optimization problem would then be formulated

$$\text{minimize} \quad J$$

$$\text{subject to} \quad \dot{\underline{x}}_1 = \underline{x}_2 \qquad , \qquad \underline{x}_1(0) = \underline{x}_1^o$$

$$\dot{\underline{x}}_2 = -a_o\underline{x}_1 - a_1\underline{x}_2 + \underline{u}, \qquad \underline{x}_2(0) = \underline{x}_2^o$$

$$\text{and} \qquad x_{11}^2 + x_{12}^2 + x_{13}^2 - 1 < 0$$

$$N\underline{\sigma} - \underline{c} \leq 0$$

where the desired performance measure is specified for J and the appropriate transformation from $\underline{\sigma}$ to $\underline{u}$ is substituted in the last constraint. Whether a closed form expression for u can be found is questionable at this time, but such a solution would be extremely helpful.

Until the problem described above is solved in closed form, techniques such as those presented in Chapter Five, and improvements thereof, must be employed to compensate for bounded inputs. The Optimal

Decision Strategy may be applied during saturation in conjunction with reinitialization of the closed form unconstrained solutions. Some improvement in performance during actuator saturation may occur, but the cost of this improvement is that of running the quadratic programming problem between sample times.

This cost might be significantly reduced by the still experimental analog quadratic minimization circuitry described by Chua and Lin in [30]. A possible control system could then be implemented as follows. The reinitialized, closed form, unconstrained solutions would be generated by a microprocessor. Linear system inputs would be digitally transformed to voltages proportional to the desired torques. When the commanded torques require actuator saturation, these voltages would be sent to the analog quadratic minimization circuitry which would instantaneously output the optimal correction signals. The corrected voltages would then drive the power electronics of the actuators. Such a hybrid approach could sharply reduce the on-line computational burden decried by Chief Scientist of the Air Force, Dr. Allen Stubberud [31].

Finally, perhaps the most interesting (and most pressing) extension of the equivalent linear approach to spacecraft attitude control is the application to flexible spacecraft. Throughout this discussion a rigid body has been assumed. As solar panels are added to provide a renewable energy source, however, this rigid body assumption becomes obsolete. The models will be more complex, and the interfaces may become extremely complicated, but the advantages of control design for a linear model warrant an attempt at applying linearizing transformations to flexible spacecraft control.

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APPENDIX A

STATE TRANSITION MATRIX AND FEEDBACK COEFFICIENTS FOR THE SINGLE-AXIS SPACECRAFT ATTITUDE CONTROL PROBLEM SOLVED IN NONLINEAR SPACE

Using the same terminology as is used in section 1.3, the state transition matrix can be expressed in the form

$$e^{\tilde{\Phi} t} = \begin{bmatrix} a_1 e^{-p_1 t} + a_2 e^{-p_2 t} & b_1 e^{-p_1 t} + b_2 e^{-p_2 t} \\ c_1 e^{-p_1 t} + c_2 e^{-p_2 t} & d_1 e^{-p_1 t} + d_2 e^{-p_2 t} \end{bmatrix}$$

The coefficients of t in the exponentials, p_1 and p_2, are determined by the moment of inertia I_o and the state penalty weights q_1, q_2 by factoring

$$s^2 + \frac{1}{I_o} \sqrt{I_o \sqrt{q_1} + q_2}\; s + \frac{1}{2I_o} \sqrt{q_1} = (s+p_1)\ (s+p_2). \qquad \text{(A.1)}$$

Coefficients a_1 through d_2 can be determined by partial fraction expansion while the state transition matrix is in the Laplace domain. All the above coefficients may be either real or complex.

Torque input for this problem is determined by feedback of terms of successively higher order in the state variables,

$$\tau = [\delta_1 \gamma + \delta_2 \omega] + [\delta_3 \gamma^3 + \delta_4 \gamma^2 \omega + \delta_5 \gamma \omega^2 + \delta_6 \omega^3] + \cdots.$$

The first and third order coefficients are listed here.

$$\delta_1 = - \sqrt{q_1}$$

$$\delta_2 = - \sqrt{I_o \sqrt{q_1} + q_2}$$

$$\delta_3 = - \frac{1}{2I_o} \delta_1\delta_2 \Big[\frac{1}{4p_1} (a_1^3d_1+3a_1^2b_1c_1) + \frac{1}{3p_1+p_2} (a_1^3d_2+3a_1^2a_2d_1+3a_1^2b_1c_2$$

$$+ 3a_1^2b_2c_1+6a_1a_2b_1c_1) + \frac{1}{2p_1+2p_2} (3a_1^2a_2d_2+3a_1a_2^2d_1+3a_1^2b_2c_2$$

$$+ 6a_1a_2b_1c_2+6a_1a_2b_1c_2+6a_1a_2b_2c_1+3a_2^2b_1c_1)$$

$$+ \frac{1}{p_1+3p_2} (6a_1a_2b_2c_2 \quad a_2^2b_1c_2+3a_2^2b_2c_1+3a_1a_2^2d_2+a_2^3d_1)$$

$$+ \frac{1}{4p_2} (3a_2^2b_2c_2+a_2^3d_2)\Big]$$

$$+\frac{1}{4}\delta_1\Big[\frac{1}{4p_1}(2a_1^2c_1d_1+2a_1b_1c_1^2) + \frac{1}{3p_1+p_2} (2a_1^2c_1d_2+2a_1^2c_2d_1+4a_1a_2c_1d_1$$

$$+ 4a_1b_1c_1c_2+2a_2b_1c_1^2+2a_1b_2c_1^2) + \frac{1}{2p_1+2p_2} (2a_1^2c_2d_2+4a_1a_2c_1d_2$$

$$+ 4a_1a_2c_2d_1+2a_2^2c_1d_1+2a_2b_2c_1^2+4a_2b_1c_1c_2+4a_1b_2c_1c_2+2a_1b_1c_2^2)$$

$$+ \frac{1}{p_1+3p_2}(2a_2^2c_1d_2+2a_2^2c_2d_1+4a_1a_2c_2d_2+2a_2b_1c_2^2+2a_1b_2c_2^2+4a_2b_2c_1c_2)$$

$$+ \frac{1}{4p_2} (2a_2^2c_2d_2+2a_2b_2c_2^2)\Big]$$

$$
\begin{aligned}
\delta_4 = &- \frac{1}{I_o}\delta_1\delta_2 \,\Big[\frac{1}{4p_1}\,(3a_1^2b_1d_1+3a_1b_1^2c_1) + \frac{1}{3p_1+p_2}\,(3a_1^2b_1d_2+3a_1^2b_2d_1+6a_1a_2b_1d_1 \\
&+ 3a_1b_1^2c_2+3a_2b_1^2c_1+6a_1b_1b_2c_1) + \frac{1}{2p_1+2p_2}\,(3a_1^2b_2d_2+6a_1a_2b_1d_2 \\
&+ 6a_1a_2b_2d_1+3a_2^2b_1d_1+3a_2b_1^2c_2+6a_1b_1b_2c_2+6a_2b_1b_2c_1+3a_1b_2^2c_1) \\
&+ \frac{1}{p_1+3p_2}\,(6a_1a_2b_2d_2 + 3a_2^2b_1d_2+3a_2^2b_2d_1+6a_2b_1b_2c_2+3a_1b_2^2c_2 \\
&+ 3a_2b_2^2c_1) + \frac{1}{4p_2}\,(3a_2^2b_2d_2+3a_2b_2^2c_2)\Big] \\
&+ \frac{1}{2}\,\delta_1\,\Big[\frac{1}{4p_1}\,(a_1^2d_1^2+4a_1b_1c_1d_1+b_1^2c_1^2) + \frac{1}{3p_1+p_2}\,(2a_1^2d_1d_2+2a_1a_2d_1^2 \\
&+ 4a_1b_1c_1d_2+4a_1b_1c_2d_1+4a_1b_2c_1d_1+4a_2b_1c_1d_1+2b_1^2c_1c_2+2b_1b_2c_1^2) \\
&+ \frac{1}{2p_1+2p_2}\,(a_1^2d_2^2+4a_1a_2d_1d_2+a_2^2d_1^2+4a_1b_1c_2d_2+4a_1b_2c_1d_2+4a_1b_2c_2d_1 \\
&+ 4a_2b_1c_1d_2+4a_2b_1c_2d_1+4a_2b_2c_1d_1+b_1^2c_2^2+4b_1b_2c_1c_2+b_2^2c_1^2) \\
&+ \frac{1}{p_1+3p_2}\,(2a_1a_2d_2^2+2a_2^2d_1d_2+4a_1b_2c_2d_2+4a_2b_1c_2d_2+4a_2b_2c_1d_2 \\
&+ 4a_2b_2c_2d_1+2b_1b_2c_2^2+2b_2^2c_1c_2) + \frac{1}{4p_2}\,(a_2^2d_2^2+4a_2b_2c_2d_2+b_2^2c_2^2)\Big]
\end{aligned}
$$

$$
\delta_5 = - \frac{3}{2I_o}\delta_1\delta_2 \,\Big[\frac{1}{4p_1}\,(3a_1b_1^2d_1+b_1^3c_1) + \frac{1}{3p_1+p_2}(3a_1b_1^2d_2+3a_2b_1^2d_1+6a_1b_1b_2d_1
$$

$$+ b_1^3c_2+3b_1^2b_2c_1) + \frac{1}{2p_1+2p_2} (3a_2b_1^2d_2+6a_1b_1b_2d_2+6a_2b_1b_2d_1$$

$$+ 3a_1b_2^2d_1+3b_1^2b_2c_2+3b_1b_2^2c_1) + \frac{1}{p_1+3p_2} (6a_2b_1b_2d_2+3a_1b_2^2d_2$$

$$+ 3a_2b_2^2d_1+3b_1b_2^2c_2+b_2^3c_1) + \frac{1}{4p_2} (3a_2b_2^2d_2+b_2^3c_2)]$$

$$+ \frac{3}{4}\delta_1 [\frac{1}{4p_1} (2a_1b_1d_1^2+2b_1^2c_1d_1) + \frac{1}{3p_1+p_2} (2a_2b_1d_1^2+2a_1b_2d_1^2+4a_1b_1d_1d_2$$

$$+ 2b_1^2c_1d_2+2b_1^2c_2d_1+4b_1b_2c_1d_1) + \frac{1}{2p_1+2p_2} (2a_2b_2d_1^2+4a_2b_1d_1d_2$$

$$+ 4a_1b_2d_1d_2+2a_1b_1d_2^2+2b_1^2c_2d_2+4b_1b_2c_1d_2+4b_1b_2c_2d_1+2b_2^2c_1d_1)$$

$$+ \frac{1}{p_1+3p_2} (4a_2b_2d_1d_2+2a_2b_1d_2^2+2a_1b_2d_2^2+4b_1b_2c_2d_2+2b_2^2c_1d_2$$

$$+ 2b_2^2c_2d_1) + \frac{1}{4p_2} (2a_2b_2d_2^2+2b_2^2c_2d_2)]$$

$$\delta_6 = - \frac{2}{I_o} \delta_1\delta_2 [\frac{1}{4p_1} (b_1^3d_1) + \frac{1}{3p_1+p_2} (b_1^3d_2+3b_1^2b_2d_1) + \frac{1}{2p_1+2p_2} (3b_1^2b_2d_2$$

$$+ 3b_1b_2^2d_1) + \frac{1}{p_1+3p_2} (3b_1b_2^2d_2+b_2^3d_1) + \frac{1}{4p_2} (b_2^3d_2)]$$

$$+ \delta_1 [\frac{1}{4p_1} (b_1^2d_1^2) + \frac{1}{3p_1+p_2} (2b_1^2d_1d_2+2b_1b_2d_1^2) + \frac{1}{2p_1+2p_2} (b_1^2d_2^2$$

$$+ 4b_1b_2d_1d_2+b_2^2d_1^2) + \frac{1}{p_1+3p_2} (2b_1b_2d_2^2+2b_2^2d_1d_2) + \frac{1}{4p_2} (b_2^2d_2^2)]$$

APPENDIX B

LINEAR-QUADRATIC OPTIMIZATION SOLUTIONS FOR LINEAR SYSTEM EQUIVALENTS OF THE SPACECRAFT ATTITUDE CONTROL PROBLEM

Detailed solutions for the five examples presented in Chapter 3 are given here. The general problem statement is

$$\text{minimize} \quad J = \frac{1}{2}\underline{x}^T(t_f)P_f\underline{x}(t_f) + \int_o^{t_f} \left(\frac{1}{2}\underline{x}^T(t)Q\underline{x}(t) + \frac{1}{2}\underline{u}^T(t)\underline{u}(t)\right)dt$$

$$\text{subject to } \dot{\underline{x}}(t) = F\underline{x}(t)+G\underline{u}(t), \quad \underline{x}(0) = \underline{x}^o, \quad \underline{x}(t_f) = \underline{0}.$$

Solution is by use of Pontryagin's Minimization Principle [32], and gives $\underline{u} = - R^{-1}G^TP\underline{x}$

where $\dot{P} = - PF - F^TP + PGR^{-1}G^TP - Q, \quad P(t_f) = P_f.$

Since the system equations in each case are decoupled, the problems are solved for the scalar system, i.e., $\underline{x} = (x_1,x_2)^T$, $\underline{u} = u$.

An initial time other than zero will only affect the determination of coefficients in the analytic expressions for the state variables. A desired final state $\underline{x}^f$ other than zero can be achieved by replacing $\underline{x}(t_f)$ by $\underline{x}(t_f) - \underline{x}^f$ in the finite final time problems, or by replacing $\underline{x}(t)$ by $\underline{x}(t) - \underline{x}^f$ in the infinite final time problems. The five examples presented in Chapter 3 are identified, in the same order, in

App. B.1 through B.5 by whether final time is finite or infinite, whether the control penalty is on $||\underline{u}||$ or $||\dot{\underline{u}}||$, and whether the state equations describe an integrator or damped system.

APPENDIX B.1

FINITE TIME, PENALTY ON $||\underline{u}||$, INTEGRATOR SYSTEM

The problem is stated

$$\text{minimize } J = \frac{1}{2}\, p_1 x_1^2(t_f) + \frac{1}{2} p_2 x_2^2(t_f) + \frac{1}{2}\int_{t_o}^{t_f} u^2(t)dt$$

$$\text{subject to} \quad \dot{x}_1 = x_2 \qquad x_1(0) = x_1^o \qquad x_1(t_f) = 0$$

$$\dot{x}_2 = u \qquad x_2(0) = x_2^o \qquad x_2(t_f) = 0.$$

System and penalty matrices are defined

$$F = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad G = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad P_f = \begin{bmatrix} p_1 & 0 \\ 0 & p_2 \end{bmatrix}, \qquad Q = 0, \qquad R = 1.$$

Solution of the matrix Ricatti equation

$$\dot{P} = -PF - F^T P + PGR^{-1}G^T P - Q, \qquad P(t_f) = P_f$$

is simplified by eliminating Q, since it is 0, and finding the differential equation for $Z = (P^{-1})$. Using the relationship $\dot{Z} = -P^{-1}\dot{P}P^{-1}$,

$$\dot{Z} = FZ + ZF^T - GG^T, \qquad Z(t_f) = P_f^{-1}.$$

Since exact terminal control is desired, p_1 and p_2 are infinite, therefore $P_f^{-1} = \begin{bmatrix} 1/p_1 & 0 \\ 0 & 1/p_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

Defining $Z = \begin{bmatrix} z_{11} & z_{12} \\ z_{12} & z_{22} \end{bmatrix}$, three differential equations must be solved,

APPENDIX B.5

INFINITE TIME, PENALTY ON $||\underline{u}||$, D.'`'ED SYSTEM

The problem is stated

$$\text{minimize } J = \frac{1}{2}\int_0^\infty (q_1x_1^2(t) + q_2x_2^2(t) + u^2(t))dt$$

$$\text{subject to} \quad \dot{x}_1 = x_2 \qquad\qquad x_1(0) = x_1^o$$

$$\dot{x}_2 = -a_ox_1-a_1x_2+u \qquad x_2(0) = x_2^o.$$

System and penalty matrices are now

$$F = \begin{bmatrix} 0 & 1 \\ -a_o & -a_1 \end{bmatrix}, \quad G = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad P_f = 0, \quad Q = \begin{bmatrix} q_1 & 0 \\ 0 & q_2 \end{bmatrix}, \quad R = 1,$$

and the algebraic equations resulting from the Ricatti equation are

$$p_{12}^2 = -2a_op_{12} + q_1$$

$$p_{22}^2 = -2a_1p_{22}+2p_{12}+q_2$$

$$p_{11} = a_1p_{12} + a_op_{22} + p_{12}p_{22},$$

or

$$p_{12} = -a_o \pm \sqrt{a_o^2+q_1}$$

$$p_{22} = -a_1 \pm \sqrt{a_1^2+2p_{12}+q_2}$$

$$p_{11} = a_1p_{12}+a_op_{22}+p_{12}p_{22}.$$

$$c_2 = -(2\alpha x_2^o + u^o)/\alpha^2$$

$$c_3 = (\alpha^2 x_1^o + 2\alpha x_2^o + u^o)/\alpha^2.$$

$$c_3 = x_1^o.$$

When arbitrary initial time, t_o, replaces 0 in the performance index, equations (B.4.2) remain unchanged, but the expressions for the constants become

$$c_1 = \frac{1}{2}\left[\alpha^2 x_1(t_o) - 2\alpha x_2(t_o) + u(t_o)\right] e^{-\alpha t_o}$$

$$c_2 = \left[(-\alpha^2 t_o - \alpha)x_1(t_o) + (2\alpha t_o + 1)x_2(t_o) - t_o u(t_o)\right]e^{-\alpha t_o}$$

$$c_3 = \frac{1}{2}\left[(\alpha^2 t_o^2 + 2\alpha t_o + 2)x_1(t_o) - (2\alpha t_o^2 + 2t_o)x_2(t_o) + t_o^2 u(t_o)\right]e^{-\alpha t_o}.$$

As in problem 2, solution of the detumbling equation reduces to the case with $||\underline{u}||$ penalized by letting x_2, u, and $u^{(1)}$ replace x_1, x_2, and u. Setting $q_1 = 0$, the differential equation for $x_1(t)$ then becomes

$$\dddot{x}_1 + \sqrt{2\sqrt{q_2} + q_3}\;\ddot{x}_1 + \sqrt{q_2}\;\dot{x}_1 = 0.$$

If q_2 and q_3 are chosen such that $q_3 = 2\sqrt{q_2}$, then

$$x_1(t) = c_1 t e^{-\alpha t} + c_2 e^{-\alpha t} + c_3$$

$$x_2(t) = -\alpha c_1 t e^{-\alpha t} + (c_1 - \alpha c_2)e^{-\alpha t}$$

$$u(t) = \alpha^2 c_1 t e^{-\alpha t} - (2\alpha c_1 - \alpha^2 c_2)e^{-\alpha t}$$

where $\quad c_1 = -(\alpha x_2^o + u^o)/\alpha$

$$p_{23}^4 - 2q_2p_{23}^2 - 8q_1p_{23} + (q_2^2 - 4q_1q_3) = 0.$$

With q_1, q_2, q_3 given, values can be found for p_{13}, p_{23}, and p_{33}, in that order. The rest of the entries follow, but care must be exercised in choosing roots for p_{13}, p_{23}, and p_{33} that result in a positive definite matrix P. The feedback expression for $u^{(1)}(t)$ in general is

$$\begin{aligned} u^{(1)}(t) &= -R^{-1}G^TPx_1(t) \\ &= -\ p_{13}x_1(t) - p_{23}x_2(t) - p_{33}u(t). \end{aligned} \qquad \text{(B.4.1)}$$

A differential equation for $x_1(t)$ is obtained

$$\dddot{x}_1 + p_{33}\ddot{x}_1 + p_{23}\dot{x}_1 + p_{13}x = 0.$$

For the example in Chapter 3, the penalties are chosen (quite premeditated) so that $p_{13} = \sqrt{q_1} = \alpha^3$, $p_{23} = 3\alpha^2$, and $p_{33} = 3\alpha$. Then the response is given by

$$\begin{aligned} x_1(t) &= c_1t^2e^{-\alpha t} + c_2te^{-\alpha t} + c_3e^{-\alpha t} \\ x_2(t) &= -\ \alpha c_1t^2e^{-\alpha t} + (2c_1-\alpha c_2)te^{-\alpha t} + (c\_\alpha c_3)e^{-\alpha t} \qquad \text{(B.4.2)} \\ u(t) &= \alpha^2c_1t^2e^{-\alpha t} - (4\alpha c_1-\alpha^2c_2)te^{-\alpha t} + (2c_1-2\alpha c_2+\alpha^2c_3)e^{-\alpha t} \end{aligned}$$

where $\quad c_1 = \frac{1}{2}(\alpha^2x_1^o + 2\alpha x_2^o + u^o)$

$$c_2 = \alpha x_1^o + x_2^o$$

APPENDIX B.4

INFINITE TIME, PENALTY ON $||\dot{\underline{u}}||$, INTEGRATOR SYSTEM

The problem is stated

$$\text{minimize } J = \frac{1}{2}\int_0^\infty (q_1 x_1^2(t) + q_2 x_2^2(t) + q_3 u^2(t) + (u^{(1)}(t))^2)dt$$

$$\text{subject to} \quad \dot{x}_1 = x_2 \qquad x_1(0) = x_1^0$$

$$\dot{x}_2 = u \qquad x_2(0) = x_2^0$$

$$\dot{u} = u^{(1)} \qquad u(0) = u^0.$$

System and penalty matrices are defined

$$F = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad G = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad P_f = 0, \quad Q = \begin{bmatrix} q_1 & 0 & 0 \\ 0 & q_2 & 0 \\ 0 & 0 & q_3 \end{bmatrix}, \quad R = 1.$$

The steady state Ricatti equation gives six nonlinear coupled algebraic equations

$$p_{13}^2 = q_1$$
$$p_{23}^2 = 2p_{12} + q_2$$
$$p_{33}^2 = 2p_{23} + q_3$$
$$p_{11} = p_{13}p_{23}$$
$$p_{12} = p_{13}p_{33}$$
$$p_{22} = p_{23}p_{33} - p_{13}.$$

Proper choice of substitutions leads to an uncoupled equation in p_{23},

$$c_2 = x_1^o$$

and $$\alpha = \sqrt{2\sqrt{q_1} + q_2} = \sqrt[4]{q_1}.$$

For the detumbling only maneuver, $q_1=0$ and the resulting expressions are

$$x_1(t) = x_1^o + \frac{1}{\sqrt{q_2}} x_2^o (1 - e^{-\sqrt{q_2}t})$$

$$x_2(t) = x_2^o e^{-\sqrt{q_2}t}$$

$$u(t) = -\sqrt{q_2}\, x_2^o e^{-\sqrt{q_2}t}.$$

The combination of the two conditions implies $p_{22} > 0$, and condition 1) with $p_{11} = p_{12}p_{22}$ implies $p_{12} > 0$. Then

$$p_{12} = \sqrt{q_1}$$

$$p_{22} = \sqrt{2\sqrt{q_1} + q_2}.$$

Feedback control is given as

$$u(t) = -R^{-1}G^TPx(t)$$

$$= -\sqrt{q_1}\, x_1(t) - \sqrt{2\sqrt{q_1} + q_2}\, x_2(t) \tag{B.3.1}$$

which yields a differential equation for $x_1(t)$

$$\ddot{x}_1 + \sqrt{2\sqrt{q_1} + q_2}\, \dot{x}_1 + \sqrt{q_1}\, x_1 = 0.$$

Selection of the state penalities q_1 and q_2 determines the type of response exhibited by x_1. A critically damped result occurs if $q_2 = 2\sqrt{q_1}$. In that case

$$x_1(t) = c_1te^{-\alpha t} + c_2e^{-\alpha t}$$

$$x_2(t) = -\alpha c_1te^{-\alpha t} + (c_1 - \alpha c_2)e^{-\alpha t} \tag{B.3.2}$$

$$u(t) = \alpha^2c_1te^{-\alpha t} - (2\alpha c_1 - \alpha^2c_2)e^{-\alpha t}$$

where $\quad c_1 = \alpha x_1^o + x_2^o$

APPENDIX B.3

INFINITE TIME, PENALTY ON $||\underline{u}||$, INTEGRATOR SYSTEM

The problem is stated

$$\text{minimize } J = \frac{1}{2}\int_0^\infty (q_1 x_1^2(t) + q_2 x_2^2(t) + u^2(t))dt$$

$$\text{subject to} \quad \dot{x}_1 = x_2 \qquad x_1(0) = x_1^o$$

$$\dot{x}_2 = u \qquad x_2(0) = x_2^o.$$

System and penalty matrices are defined

$$F = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad G = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad P_f = 0, \quad Q = \begin{bmatrix} q_1 & 0 \\ 0 & q_2 \end{bmatrix}, \quad R = 1.$$

Now the Ricatti equation becomes

$$0 = - PF - F^T P + PGR^{-1}G^T P - Q$$

and gives

$$p_{12}^2 = q_1$$

$$p_{22}^2 = 2p_{12} + q_2$$

$$p_{11} = p_{12}p_{22}.$$

For the resulting gain matrix to be positive definite, two conditions must hold

$$1) \quad p_{11} > 0 \qquad \text{and} \qquad 2) \quad p_{11}p_{22} - p_{12}^2 > 0.$$

$$c_1 = 6t_f^{-5}x_1^o + 3t_f^{-4}x_2^o + 0.5t_f^{-3}u^o$$

$$c_2 = -15t_f^{-4}x_1^o - 7t_f^{-3}x_2^o - 1.0t_f^{-2}u^o$$

$$c_3 = 10t_f^{-3}x_1^o + 4t_f^{-2}x_2^o + 0.5t_f^{-1}u^o.$$

When an arbitrary initial time, t_o, is used in place of 0, equations (B.2.2) remain the same, but t_f is replaced by (t_f-t_o) in the expressions for c_1, c_2, and c_3. Detumbling with arbitrary terminal orientation reduces this optimization problem to precisely the form of the one solved in App. B.1 where x_2, u, and $u^{(1)}$ replace x_1, x_2, and u respectively. Then

$$x_1(t) = c_1 - \frac{1}{4}c_2(t_f-t)^4 - \frac{1}{3}c_3(t_f-t)^3$$

$$x_2(t) = c_2(t_f-t)^3 + c_3(t_f-t)^2$$

$$u(t) = -3c_2(t_f-t)^2 - 2c_3(t_f-t)$$

and

$$c_1 = x_1^o + \frac{1}{2}t_f x_2^o + \frac{1}{12}t_f^2 u^o$$

$$c_2 = -2t_f^{-3}x_2^o - t_f^{-2}u^o$$

$$c_3 = 3t_f^{-2}x_2^o + t_f^{-1}u^o.$$

Again note the reduction by one of the order of the polynomial expressions for x_1, x_2, and u when the specification of the terminal state is dropped.

$$Z = \begin{bmatrix} \frac{1}{20}(t_f-t)^5 & -\frac{1}{8}(t_f-t)^4 & \frac{1}{6}(t_f-t)^3 \\ -\frac{1}{8}(t_f-t)^4 & \frac{1}{3}(t_f-t)^3 & -\frac{1}{2}(t_f-t)^2 \\ \frac{1}{6}(t_f-t)^3 & -\frac{1}{2}(t_f-t)^2 & (t_f-t) \end{bmatrix}$$

and

$$P = Z^{-1} = \begin{bmatrix} 720(t_f-t)^{-5} & 360(t_f-t)^{-4} & 60(t_f-t)^{-3} \\ 360(t_f-t)^{-4} & 192(t_f-t)^{-3} & 36(t_f-t)^{-2} \\ 60(t_f-t)^{-3} & 36(t_f-t)^{-2} & 9(t_f-t)^{-1} \end{bmatrix}.$$

The closed loop expression for $u^{(1)}$ is then

$$u^{(1)}(t) = -60(t_f-t)^{-3}x_1(t) - 36(t_f-t)^{-2}x_2(t) - 9(t_f-t)^{-1}u(t). \quad \text{(B.2.1)}$$

A differential equation for $x_1(t)$ is found

$$\dddot{x}_1 + 9(t_f-t)^{-1}\ddot{x}_1 + 36(t_f-t)^{-2}\dot{x}_1 + 60(t_f-t)x_1 = 0$$

which is again an Euler differential equation. Assuming $x_1(t) = (t_f-t)^r$, a cubic equation for r is found yielding r=5, r=4, and r=3. Then

$$\begin{aligned} x_1(t) &= c_1(t_f-t)^5 + c_2(t_f-t)^4 + c_3(t_f-t)^3 \\ x_2(t) &= -5c_1(t_f-t)^4 - 4c_2(t_f-t)^3 - 3c_3(t_f-t)^2 \\ u(t) &= 20c_1(t_f-t)^3 + 12c_2(t_f-t)^2 + 6c_3(t_f-t) \end{aligned} \quad \text{(B.2.2)}$$

where c_1, c_2, c_3 are found by evaluating (B.2.2) at the boundary conditions

APPENDIX B.2

FINITE TIME, PENALTY ON $||\dot{u}||$, INTEGRATOR SYSTEM

The problem is stated

$$\text{minimize } J = \frac{1}{2}p_1 x_1^2(t_f) + \frac{1}{2}p_2 x_2^2(t_f) + \frac{1}{2}p_3 u^2(t_f) + \frac{1}{2}\int_0^{t_f} (u^{(1)}(t))^2 dt$$

$$\text{subject to} \quad \dot{x}_1 = x_2 \qquad x_1(0) = x_1^o \qquad x_1(t_f) = 0$$

$$\dot{x}_2 = u \qquad x_2(0) = x_2^o \qquad x_2(t_f) = 0$$

$$\dot{u} = u^{(1)} \qquad u(0) = u^o \qquad u(t_f) = 0.$$

System and penality matrices are defined

$$F = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad G = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad P_f = \begin{bmatrix} p_1 & 0 & 0 \\ 0 & p_2 & 0 \\ 0 & 0 & p_3 \end{bmatrix}, \quad Q = 0, \quad R = 1.$$

Using the same procedure to solve the matrix Ricatti equation as described in App. B.1, six differential equations are found

$$\dot{z}_{11} = 2z_{12} \qquad z_{11}(t_f) = 0$$

$$\dot{z}_{12} = z_{22}+z_{13} \qquad z_{12}(t_f) = 0$$

$$\dot{z}_{13} = z_{23} \qquad z_{13}(t_f) = 0$$

$$\dot{z}_{22} = 2z_{23} \qquad z_{22}(t_f) = 0$$

$$\dot{z}_{23} = z_{33} \qquad z_{23}(t_f) = 0$$

$$\dot{z}_{33} = -1 \qquad z_{33}(t_f) = 0.$$

Solving these equations with the boundary conditions specified,

equation becomes a scalar equation $\dot{p} = p^2$, $p(t_f) = p_2$. Then $\dot{z} = d(1/p)/dt = -(\dot{p}/p^2) = -1$, $z(t_f) = 0$. The solution is

$$z = (t_f - t)$$

$$p = (t_f - t)^{-1}$$

$$u(t) = -\,(t_f - t)^{-1} x_2(t).$$

Finding analytic expressions for x_1, x_2, and u in the same manner as earlier,

$$x_1(t) = x_1^o + 0.5 t_f x_2^o - 0.5 t_f^{-1} x_2^o (t_f - t)^2$$

$$x_2(t) = t_f^{-1} x_2^o (t_f - t)$$

$$u(t) = -\,t_f^{-1} x_2^o.$$

An interesting result is that when the final orientation is specified, the control u is described by a ramp; but when the final orientation is unspecified, the control is a step input.

$$(t_f-t)^2\ddot{x}_1 + 4(t_f-t)\dot{x}_1 + 6x_1 = 0.$$

This Euler differential equation is solved by assuming a solution of the form $x_1(t) = (t_f-t)^r$ and substituting [33]. An algebraic equation in r results and roots r=3 and r=2 are found. Therefore

$$x_1(t) = c_1(t_f-t)^3 + c_2(t_f-t)^2.$$

Differentiating,

$$x_2(t) = -3c_1(t_f-t)^2 - 2c_2(t_f-t).$$

Coefficients c_1 and c_2 are determined by evaluating $x_1(t)$ and $x_2(t)$ at t=0 and equating those expressions to the initial conditions. Finally, the open loop control law and resulting state variable histories are given by

$$\begin{aligned} x_1(t) &= c_1(t_f-t)^3 + c_2(t_f-t)^2 \\ x_2(t) &= -3c_1(t_f-t)^2 - 2c_2(t_f-t) \\ u(t) &= 6c_1(t_f-t) + 2c_2 \end{aligned} \tag{B.1.2}$$

where

$$\begin{aligned} c_1 &= -2t_f^{-3}x_1^o - t_f^{-2}x_2^o \\ c_2 &= 3t_f^{-2}x_1^o + t_f^{-1}x_2^o. \end{aligned}$$

If a detumbling maneuver is desired and the final spacecraft orientation is arbitrary, p_1 can be set to zero. The system equation is simply $\dot{x}_2 = u$ with boundary constraints $x_2(0) = x_2^o$, $x_2(t_f) = 0$, and the Ricatti

$$\dot{z}_{11} = 2z_{12} \qquad z_{11}(t_f) = 0$$

$$\dot{z}_{12} = z_{22} \qquad z_{12}(t_f) = 0$$

$$\dot{z}_{22} = -1 \qquad z_{22}(t_f) = 0.$$

This results in

$$Z = \begin{bmatrix} \frac{1}{3}(t_f-t)^3 & -\frac{1}{2}(t_f-t)^2 \\ -\frac{1}{2}(t_f-t)^2 & (t_f-t) \end{bmatrix}$$

and

$$P = Z^{-1} = \begin{bmatrix} 12(t_f-t)^{-3} & 6(t_f-t)^{-2} \\ 6(t_f-t)^{-2} & 4(t_f-t)^{-1} \end{bmatrix}.$$

Then the closed loop control law is

$$u(t) = -R^{-1}G^TP(t)x(t)$$

$$= -6(t_f-t)^{-2}x_1(t) - 4(t_f-t)^{-1}x_2(t). \qquad \text{(B.1.1)}$$

From the system equations,

$$\ddot{x}_1 = u = -6(t_f-t)^{-2}x_1 - 4(t_f-t)^{-1}x_2$$

a differential equation for $x_1(t)$ is found

$$\ddot{x}_1 + 4(t_f-t)^{-1}\dot{x}_1 + 6(t_f-t)^{-2}x_1 = 0$$

or

Since one desired result of introducing a_o and a_1 in the model is to achieve a stable system, both must be nonnegative. A matrix Ricatti equation has one symmetric positive definite solution. Positive definite for P implies $p_{11}p_{22} - p_{12}^2 > 0$. The smaller the magnitude of p_{12}, the more positive the determinant of P. Therefore, the positive definite solution of P must be the one associated with the smaller magnitude of p_{12}, and $p_{12} = -a_o + \sqrt{a_o^2+q_1}$. From the discussion in App. B.3, $p_{22}>0$; hence, $p_{22} = -a_1 + \sqrt{a_1^2+2p_{12}+q_2}$. With these results, the expression for p_{11} can be found, but that expression is not needed in the solution of this problem. Positive definiteness has already been guaranteed, for p_{12} and p_{22} are suitably chosen (and are both positive), and the expression for p_{11} only involves nonnegative terms. Therefore $p_{11}>0$. The closed loop control law is given by

$$\begin{aligned} u(t) &= -R^{-1}G^TP\underline{x}(t) \\ &= -p_{12}x_1(t) - p_{22}x_2(t) \qquad \text{(B.5.1)} \\ &= (a_o - \sqrt{a_o^2+q_1})x_1(t) + (a_1 - \sqrt{a_1^2-2a_o+2\sqrt{a_o^2+q_1}+q_2})x_2(t) \end{aligned}$$

which is independent of p_{11}.

This control law yields the differential equation for $x_1(t)$

$$\ddot{x}_1 + (a_1+p_{22})\dot{x}_1 + (a_o+p_{12})x_1 = 0.$$

As before, choice of q_1 and q_2 along with a_o and a_1 determine the type of response displayed by $x_1(t)$. If $a_1^2-2a_o-2\sqrt{a_o^2+q_1}+q_2=0$, then $x_1(t)$ takes the form

$$x_1(t) = c_1te^{-\alpha t} + c_2e^{-\alpha t}$$

where $\alpha = \frac{1}{2}\sqrt{a_1^2-2a_o+2\sqrt{a_o^2+q_1}+q_2}$. Then expressions for $x_2(t)$ and $u(t)$ are identical to those found in App. B.3.

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